# A Bi-Robust Test for Vague Data

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## Abstract

Nonparametric statistical tests are robust to disturbances of the assumptions on the underlying distributions. However, in the presence of vague data, modelled by fuzzy sets, the problem of robustness is much more complicated. Here not only distribution-free methods but tests which are also not sensitive to the particular choice of the membership functions applied for modelling vague concepts would be desirable. A construction of such bi-robust test is suggested in the paper.

**Keywords:** fuzzy sets, IF-sets, nonparametric test, sign test, vague data.

## 1 Introduction

Very often statistical tests are categorized within the framework of parametric or nonparametric ones. Parametric test makes specific assumptions the underlying population distribution (e.g., normality, exponentiality). If these assumptions are violated it may happen that a parametric test would loose its good properties. Thus, if the assumptions are not necessarily satisfied it would be more prudent to apply **nonparametric tests** (also referred to as **distribution-free tests**) which make no such stringent assumptions on the underlying distribution. This way nonparametric tests are generally perceived as more robust to possible violations of assumptions than parametric ones.

However, if we deal with vague data which are often modelled by fuzzy sets, another view on the robustness of statistical procedures appears. Since the shape of membership functions applied for modelling vague data is generally strongly subjective one may ask about the possible influence of that shape on further decisions.

Nonparametric test for fuzzy data were considered e.g. by Grzegorzewski [6], [8], [10] and [3]. Although tests considered in these paper are distribution-free, their output may heavily depend on the membership functions which characterize fuzzy data. Thus in the present paper we suggest a modification of the well-known sign test for fuzzy data which is both distribution-free and robust to the choice of the particular form of membership function describing data. In the description of this test we make use of If-sets (also known as Atanassov tests or intuitionistic fuzzy sets) which facilitate the interpretation and make it more natural.

The paper is organized as follows: In Sec. 2 we introduce basic notation used for modelling vague data. In Sec. 3, we recall the classical sign test and in Sec. 4 we mention some information on IF-sets. Then in Sec. 5 we suggest a bi-robust modification of the sign test. Finally, in Sec. 6 we show that our modified sign test could be also applied for verifying imprecise hypotheses.

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## 2 Vague data

In many areas of human cognition people state and verify hypotheses. Hypotheses testing is also an important aspect of decision making. When the hypotheses deal with objects which are not deterministic but follow any random distribution, we have so-called statistical hypotheses testing. In the classical theory of statistical hypotheses testing all parameters of the mathematical model, i.e. data, hypotheses and requirements, should be precisely defined. However, in real life we often meet vague data, like "about twenty", "more or less between fifty and sixty" and so on. It may also happen that our data are crisp but a hypothesis is imprecise. For example we may consider a hypothesis that the mean is "about ten" or that the variance is "no greater than five". In such cases one may utilize fuzzy numbers for modelling both vague data and imprecise hypotheses.

The notion of a fuzzy number was introduced by Dubois and Prade [4]. We say that a fuzzy subset $A$ of the real line $\mathbb{R}$, with the membership function $\mu_A : \mathbb{R} \to [0,1]$, is a fuzzy number if and only if $A$ is normal (i.e. there exists an element $x_0$ such that $\mu_A(x_0) = 1$), $A$ is fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$, $\forall x_1, x_2 \in \mathbb{R}$, $\forall \lambda \in [0,1]$), $\mu_A$ is upper semicontinuous and supp$A$ is bounded, where supp$A = cl(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and $cl$ is the closure operator.

A useful notion for dealing with a fuzzy number is a set of its $\alpha-$cuts. The $\alpha-$cut of a fuzzy number $A$ is a nonfuzzy set defined as

$$A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}. \qquad (1)$$

A family $\{A_\alpha : \alpha \in (0,1]\}$ is a set representation of the fuzzy number $A$. According to the definition of a fuzzy number it is easily seen that every $\alpha$-cut of a fuzzy number is a closed interval. Hence we have $A_\alpha = [A^L_\alpha, A^U_\alpha]$, where $A^L_\alpha = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ and $A^U_\alpha = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$. A space of all fuzzy numbers will be denoted by $\mathbb{FN}(\mathbb{R})$.

To model fuzzy outcomes of a random experiment we need the notion of fuzzy random variable. The first who considered fuzzy random variables was Kwakernaak [13], [14]. Other definitions of fuzzy random variables are due to Kruse [11] or to Puri and Ralescu [15]. Here let us consider a definition similar to those of Kwakernaak and Kruse. Suppose that a random experiment is described as usual by a probability space $(\Omega, \mathbb{A}, P)$, where $\Omega$ is a set of all possible outcomes of the experiment, $\mathbb{A}$ is a $\sigma$-algebra of subsets of $\Omega$ (the set of all possible events) and $P$ is a probability measure. Then mapping $X : \Omega \to \mathbb{FN}(\mathbb{R})$ is called a fuzzy random variable if it satisfies the following properties: (a) $\{X(\alpha, \omega) : \alpha \in [0,1]\}$ is a set representation of $X(\omega)$ for all $\omega \in \Omega$, (b) for each $\alpha \in [0,1]$ both $X^L_\alpha = X^L_\alpha(\omega) = \inf X_\alpha(\omega)$ and $X^U_\alpha = X^U_\alpha(\omega) = \sup X_\alpha(\omega)$, are usual real-valued random variables on $(\Omega, \mathbb{A}, P)$.

Thus a fuzzy random variable $X$ is considered as a perception of an unknown usual random variable $V : \Omega \to \mathbb{R}$, called an *original* of $X$. Similarly $n$-dimensional fuzzy random sample $X_1, \ldots, X_n$ may be treated as a fuzzy perception of the usual random sample $V_1, \ldots, V_n$ (where $V_1, \ldots, V_n$ are independent and identically distributed crisp random variables). For more information we refer the reader e.g. to [12].

Now having a mathematical model for vague outcomes of a random experiment we may pass to hypotheses testing. Grzegorzewski [7] proposed a general method for constructing statistical tests for vague data modelled by fuzzy sets. However, another problem connected with fuzzy data comes from the fact that most of the statistical tests is based on fairly specific assumptions regarding the nature of the underlying population distribution. Stringent assumptions on distributions lead sometimes to serious difficulties even in the case of crisp data. But this difficulty is much stronger when the data are fuzzy because the justification on the legitimacy of the postulates on the underlying distribution in the presence of imprecise data is a very serious problem. In fact we still do not have satisfactory goodness-of-fit techniques for fuzzy data. As in statistics for crisp data a remedy for this problem are nonparametric methods which are by the definition distribution-free. Some nonparametric approaches for testing hypotheses with fuzzy data were suggested, like [6], [8], [10] or [3].

Unfortunately those nonparametric tests for imprecise data, although they smooth away difficulties related to the unknown underlying distribution, they strongly depend on the shape of the membership function utilized for modelling the data. This seems to be a serious drawback, especially that using fuzzy modelling we make every endeavor to be flexible yet we are still very restricted by the very choice of the precise form of the membership functions. One can easily appreciate the weight of the problem when he realizes that different persons may assign distinct membership functions to the same objects since modelling vague objects cannot be completely free from subjectivity.

One way out is to utilize defuzzification methods, like in [8]. However, as always when the defuzzification is performed, this approach might be also criticized both for the unnecessary loss of information and for too arbitrary choice of the defuzzification method. Moreover, this way we actually do not solve the problem but we replace the matter of the procedure's susceptibility to the choice of the membership function with robustness against the defuzzification method.

Presumably it would be difficult to construct such statistical procedures for vague data that disregard completely the actual shape of the membership functions applied for modelling these data. However, we may try to eliminate the impact of the particular form of membership functions as much as possible or to the acceptable degree. Such desired tests that are robust both to assumptions on the underlying distribution and for the shape of the membership functions will be called below as **bi-robust tests**.

It seems that a good starting point for our goal is an appropriate choice of a nonparametric test which would be latter generalized for fuzzy data. But what do we mean by a good test? It should be a test which is based on as much as possible general information on the data. In nonparametric statistics we can find such methods that disregard actual values of the observations and are based only on some comparisons between the data and fixed values or just between observations. The well-known sign test seems to be a good candidate for such bi-robust test for the location problem. Thus in the next section we recall briefly the classical sign test, while in Sec. 5 we suggest its generalization for fuzzy data.

## 3 The sign test

The classical tests for the location parameter - generally for the mean - are derived under the assumption that the single population is normal. Otherwise, having large sample, one may perform asymptotic tests. However, in many situations, these assumption are not satisfied. Then nonparametric tests are recommended since they often do not require almost any assumptions about specific population distribution. The well-known sign test seems to be a good nonparametric alternative to parametric tests for single population location problem. In the sign test the hypotheses concerns the median, not the mean, as a location parameter. Both the mean and the median are good measures of central tendency and they coincide for symmetric distributions. But in any population the median always exists, which is not true for the mean. Moreover, the median is more robust to outlier as an estimate of location than the mean.

Suppose a random sample of $n$ independent observations $V_1, \ldots, V_n$ is drawn from the population with unknown median $M$. The only assumption about the distribution is that the population distribution is continuous in the vicinity of $M$. We verify the null hypothesis concerning the value of the population median

$$H_0 : M = M_0 \tag{2}$$

with a corresponding one-sided or two-sided alternative on $M$.

The idea of the sign test is very simple: if the data are consistent with the hypothesized median $M_0$, on the average half of the sample observations should lie above $M_0$ and a half below. Thus if $H_0$ holds and we gather only the signs of all differences $V_i - M_0$ for $i = 1, \dots, n$, then the number of plus signs and the number of minus signs will be more or less identical. And conversely, if there is significant disproportion between the number of plus signs and the number of minus signs then we may conclude that hypothesis $H_0$ should be rejected. This is the reason why we call this test the sign test. Of course, it is not necessary to consider both the number of plus signs and the number of minus signs because they are strongly related. So the test statistic is defined as follows

$$T = \sum_{i=1}^{n} \mathbb{I}(V_i > M_0), \tag{3}$$

where $\mathbb{I}(\rho)$ denotes the indicator function such that

$$\mathbb{I}(\rho) = \begin{cases} 1 & \text{if} \quad \rho \text{ is true,} \\ 0 & \text{if} \quad \rho \text{ is false.} \end{cases}$$

As it is seen, the test statistic is just the number of plus signs among $n$ differences. The sampling distribution of $T$ is binomial with parameters $n$ and $\theta$ which is equal to 0.5 if the null hypothesis $H_0$ holds. The appropriate rejection region depends on the alternative hypothesis. For a one-sided upper-tailed alternative $H_1 : M > M_0$ we have a following decision rule:

$$\begin{array}{ll} \text{if } T \geq k_\delta & \text{then reject } H_0, \\ \text{if } T < k_\delta & \text{then accept } H_0, \end{array} \tag{4}$$

where $k_\delta$ is chosen to be the smallest integer which satisfies

$$P(T \geq k_\delta \mid H_0) = \sum_{i=k_\delta}^{n} \binom{n}{i} 0.5^n \leq \delta \tag{5}$$

and $\delta$ is an accepted significance level. Similarly, for a one-sided lower-tailed alternative $H_1 : M < M_0$ we have

$$\begin{array}{ll} \text{if } T \leq k'_\delta & \text{then reject } H_0, \\ \text{if } T > k_\delta & \text{then accept } H_0, \end{array} \tag{6}$$

where $k'_\delta = n - k_\delta$. And finally, for a two-sided alternative $H_1 : M \neq M_0$ we have

$$\begin{array}{ll} \text{if } T \geq k_{\delta/2} \text{ or } T \leq k'_{\delta/2} & \text{then reject } H_0, \\ \text{if } k'_{\delta/2} < T < k_{\delta/2} & \text{then accept } H_0. \end{array}$$

Alternatively, instead of comparing test statistic with appropriate critical values one may compute the so-called *p-value* corresponding to obtained value of test statistic $T$ (i.e. the smallest significant level at which the null hypothesis could be rejected). Then, using *p-value* the decision rule is as follows:

$$\begin{array}{ll} \text{if } p - value \leq \delta & \text{then reject } H_0, \\ \text{if } p - value > \delta & \text{then accept } H_0. \end{array} \tag{7}$$

For more information on the sign test and other nonparametric test we refer the reader to [5] or [9].

## 4 IF-sets

As it was mentioned above we use fuzzy numbers for modelling vague data. However, we will also utilize If-sets in this paper. Here we briefly recall some basic notions related to IF-sets. Let $U$ denote a universe of discourse. Then a fuzzy set $C$ in $U$ is defined as a set of ordered pairs

$$C = \{\langle x, \mu_C(x) \rangle : x \in U\}, \tag{8}$$

where $\mu_C : U \to [0, 1]$ is the membership function of $C$ and $\mu_C(x)$ is the grade of belongingness of $x$ into $C$ (see [16]). Thus automatically the grade of nonbelongingness of $x$ into $C$ is equal to $1 - \mu_C(x)$. However, in real life the linguistic negation not always identifies with logical negation. This situation is very common in natural language processing, computing with words, etc. Therefore Atanassov [1], [2] suggested a generalization of classical fuzzy set, called an intuitionistic fuzzy set. The name suggested by Atanassov is slightly misleading, because his sets have nothing in common with intuitionism known from logic. It seems that other name, e.g. incomplete fuzzy sets (which had the same abbreviation), would be even more adequate for the Atanassov sets. Thus finally, in order to avoid terminology problems, we call the Atanassov sets as IF-sets.

An IF-set $C$ in $U$ is given by a set of ordered triples

$$C = \{\langle x, \mu_C(x), \nu_C(x)\rangle : x \in U\}, \quad (9)$$

where $\mu_C, \nu_C : U \to [0,1]$ are functions such that

$$0 \leq \mu_C(x) + \nu_C(x) \leq 1 \qquad \forall x \in U. \quad (10)$$

For each $x$ the numbers $\mu_C(x)$ and $\nu_C(x)$ represent the degree of membership and degree of nonmembership of the element $x \in U$ to $C \subset U$, respectively. Of course, IF-set of a form $\{\langle x, \mu_C(x), 1 - \mu_C(x)\rangle : x \in X\}$ is equivalent to (8), i.e. each fuzzy set is a particular case of the IF-set.

For each element $x \in U$ we can compute, so called, the IF-index of $x$ in $C$ defined as follows

$$\pi_C(x) = 1 - \mu_C(x) - \nu_C(x), \quad (11)$$

which quantifies the amount of indeterminacy associated with $x$ in $C$.

## 5 Modified sign test for fuzzy data

Suppose we have a fuzzy random sample $X_1, \ldots, X_n$. When we try to apply the classical sign test directly to fuzzy data we meet immediately a serious difficulty. This is because the test statistic $T$ depends on the number of observations bigger than a value $M_0$ considered in the null hypothesis while fuzzy numbers are not linearly ordered. Since a crisp number is a special case of a fuzzy number it may happen that one would not be able to judge whether given fuzzy observation $X_i$ is greater or not than the hypothesized median $M_0$.

However, although there is no such ordering system that could univocally determine which one of two fuzzy numbers is bigger, everybody would agree to admit $X_i$ greater than $M_0$ if $\inf(\text{supp}X_i) > M_0$. Similarly, we will have no doubts to say that $M_0$ is greater than $X_i$ if $M_0 > \sup(\text{supp}X_i)$. All other cases, i.e. when $M_0 \in \text{supp}X_i$ are not so clear and different methods for ordering fuzzy number may lead to opposite conclusions. In such uncertain situations one may also try to specify a degree to which the majority relation is satisfied.

Let $G$ denote an IF-set of observations greater than $M_0$. Therefore, we have

$$G = \{\langle X_i, \mu_G(X_i), \nu_G(X_i)\rangle : i = 1, \ldots, n\}, \quad (12)$$

where $\mu_G(X_i)$ shows the degree to which observation $X_i$ is greater than $M_0$ and $\nu_G(X_i)$ represents the degree to which the above mentioned relationship is not satisfied. Thus IF-index $\pi_G(X_i) = 1 - \mu_G(X_i) - \nu_G(X_i)$ illustrates the degree of hesitancy or irresolution of the observer regarding the majority relation between $X_i$ and $M_0$.

Thus for all observations in a sample such that $\inf\text{supp}X_i > M_0$ we get $\mu_G(X_i) = 1$ and $\nu_G(X_i) = 0$. If $M_0 > \sup\text{supp}X_i$ then $\mu_G(X_i) = 0$ and $\nu_G(X_i) = 1$. For all situations when $M_0 \in \text{supp}X_i$ both $\mu_G(X_i)$ and $\nu_G(X_i)$ may assume various values which reflect how much the observer is convinced that $X_i$ greater than $M_0$ and how much he is against this statement, respectively. In particular, if the observer is neither for nor against and he wants to avoid any assessment of the degree to which the majority relation is satisfied, he may attribute to the observation following values: $\mu_G(X_i) = 0$ and $\nu_G(X_i) = 0$.

Now we may define a test statistic. As in the classical case we will try to count how many observations exceed the hypothesized median $M_0$. This task may be considered as an attempt to compute the cardinality of the set $G$. However, since $G$ is an IF-set its cardinality would not be a natural number but an interval given as follows

$$\begin{aligned} cardG &= [\sum_{i=1}^{n} \mu_G(X_i), \sum_{i=1}^{n}(1 - \nu_G(X_i))] \\ &= [\sum_{i=1}^{n} \mu_G(X_i), n - \sum_{i=1}^{n} \nu_G(X_i)]. \end{aligned} \quad (13)$$

Hence for fuzzy data the output of the test statistic is no longer a single natural number but an interval

$$\widetilde{T}(X_1, \ldots, X_n) = [T^L, T^U], \qquad (14)$$

where

$$T^L = \sum_{i=1}^{n} \mu_G(X_i), \qquad (15)$$

$$T^U = n - \sum_{i=1}^{n} \nu_G(X_i). \qquad (16)$$

To make a decision we will apply the rule based on the concept of *p-value*. However, since our modified sign test statistic is an interval, hence *p-value* corresponding to the output of the modified test would be no longer a single real number from the unit interval . Therefore, as a counterpart of the traditional *p-value* we will consider an interval $\widetilde{p}$ (also called *p-value*) given by

$$\widetilde{p} = [p^L, p^U], \qquad (17)$$

such that

$$p^L = \sum_{i=\lfloor T^L \rfloor}^{n} \binom{n}{i} 0.5^n, \qquad (18)$$

$$p^U = \sum_{i=\lceil T^U \rceil}^{n} \binom{n}{i} 0.5^n, \qquad (19)$$

where $\lfloor x \rfloor$ is the biggest integer smaller or equal to $x$, while $\lceil x \rceil$ stands for the smallest integer greater than or equal to $x$.

Using our interval *p-value* the decision rule is a little different than as in the classical case. Hence for any assumed significance level $\delta$ we have following decision rules:

$$\begin{array}{lll} \text{if} & p^U \leq \delta & \text{then reject } H_0, \\ \text{if} & p^L > \delta & \text{then accept } H_0. \end{array} \qquad (20)$$

If $p^L \leq \delta < p^U$ then our test is not decisive.

The last case when the test appears to be non-decisive simply means that given sample brings evidence neither for rejection nor for the acceptance of the hypothesis under study. In such a case one may suggest, for example, to consider a bigger sample for making the final decision. It is also worth noting that such situation when a test is not decisive is not unique even in traditional statistics for precise data - one can recall the classical Durbin-Watson test as an example.

One may also note that if all observations are crisp (not fuzzy) then the test suggested in this section reduces to the classical sign test. It means that our modified sign test for fuzzy data is a natural generalization of the original sign test.

## 6 Testing imprecise hypotheses

It is worth noting that the main idea of the test construction given above could be easily applied also for testing imprecise hypotheses. Suppose now that we verify the null hypothesis concerning the value of the population median

$$H_0 : M = \widetilde{M}_0, \qquad (21)$$

where $\widetilde{M}_0 \in \mathbb{FN}(\mathbb{R})$. This situation corresponds to imprecisely formulated hypotheses of the type "the population median is about $\widetilde{M}_0$" or "the population median is more or less $\widetilde{M}_0$".

So now we have to compare fuzzy observations $X_1, \ldots, X_n$ with fuzzy median $\widetilde{M}_0$. Keeping in mind all remarks given in the previous section we are aware of problems with determination whether given fuzzy observation is or is not greater than the hypothesized fuzzy median. However, everybody would agree to admit $X_i$ greater than $\widetilde{M}_0$ if $\inf \text{supp} X_i > \sup(\text{supp}\widetilde{M}_0)$. Similarly, we will have no doubt to say that $\widetilde{M}_0$ is greater than $X_i$ if $\inf(\text{supp}\widetilde{M}_0) > \sup(\text{supp}X_i)$. In all other cases, i.e. when $(\text{supp}\widetilde{M}_0) \cap (\text{supp}X_i) \neq \emptyset$, different methods for ordering fuzzy number may lead to opposite conclusions.

Thus, as before, one may try to specify a degree to which the majority relation is satisfied. Now IF-set $G$ given by (12) describes observations greater than $\widetilde{M}_0$, with $\mu_G(X_i)$ showing the degree to which observation $X_i$ is greater than $\widetilde{M}_0$ and $\nu_G(X_i)$ representing the degree to which the above mentioned relationship is not satisfied. Of course, for all observations in a sample such that $\inf \text{supp} X_i > \sup(\text{supp}\widetilde{M}_0)$

we get $\mu_G(X_i) = 1$ and $\nu_G(X_i) = 0$ while situations where $\inf(\text{supp}\widetilde{M}_0) > \text{supsupp}X_i$ lead to $\mu_G(X_i) = 0$ and $\nu_G(X_i) = 1$. For all other situations when $(\text{supp}\widetilde{M}_0) \cap (\text{supp}X_i) \neq \emptyset$ both $\mu_G(X_i)$ and $\nu_G(X_i)$ may assume various values which reflect how much the observer is convinced that $X_i$ greater than $\widetilde{M}_0$ and how much he is against this statement, respectively.

The test statistic and decision rules for testing imprecise hypotheses are obtained in the same way as in Sec. 5.

## 7 Conclusions

In the present paper we have proposed a modification of the classical sign test to cope with vague data modelled by fuzzy sets. The appealing feature of that modification is that we obtain a bi-robust test, i.e. a test which is both distribution-free and which does not depend so heavily on the shape of the membership functions used for modelling imprecise data. The suggested modified sign test could be also applied for testing imprecise hypotheses.

Finally, we want to stress that although this paper is dedicated to sign test modification only, using the suggested methodology one can also generalize other nonparametric tests into fuzzy environment.

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