On fuzzy successors

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Abstract

Conventional addition builds upon the ordinary successor operator. It can therefore be expected that a formal algebraic approach to fuzzy arithmetic should likewise be based on a notion of fuzzy successors, and indeed allowing for a family of fuzzy successors. In this paper we aim at formalizing the first steps towards an understanding of an algebraically formal description of fuzzy addition. **Keywords:** Fuzzy arithmetic, fuzzy successor.

1 Introduction

The paper continues investigations initiated in [5] toweards a categorical approach to fuzzification processes. Traditionally fuzzification processes on a non-empty set X are performed by applying *Zadeh Extension Principle* (ZEP) [7, 11]. In this paper we suggest to consider a formal approach to fuzzy successors. This provides groundworks for further developments that are especially interesting in arithmetic of fuzzy natural numbers. Linguistically, ZEP is usually interpreted such that any operation on X can be extended to an operation on L^X, where L is typically a completely distributive lattice and for $L = \{0, 1\}$ we write $L = \mathbf{2}$.

Example 1. Consider there are some persons, each buying a couple of wine bottles. As a result, they have all together quite many wine bottles. Clearly, 'some persons', 'a couple of wine bottles' and 'quite many wine bottles' can be modelled as L-sets on the set of natural numbers $\mathbb{N}$. The hedge 'quite many' may be considered as some kind of aggregation of 'some' and 'a couple of'.

Example 1 suggests that the arithmetic of natural numbers is not really needed, while the extension principle says that the arithmetic on $\mathbb{N}$ is defined first and then these operations are extended. Now, we recall ZEP mathematically: Let X be a non-empty set. Then, for any $A_1, \ldots, A_n \in L^X$ we have ZEP:

$$f_L(A_1, \ldots, A_n)(z) = \bigvee_{f(x_1,\ldots,x_n)=z} \left(\bigwedge_{i=1}^{n} A_i(x_i) \right)$$

where f_L is an n-ary operation on L^X as an extension of $f \colon X^n \longrightarrow X$, and $x_1, \ldots, x_n \in X$. The formula ZEP is the traditional way to define, for example, arithmetic of fuzzy natural numbers as 'arithmetic with fuzzy'. On the other hand, the study in [3] suggests that approaching arithmetic of fuzzy natural numbers as 'arithmetic with fuzzy' is counter intuitive in monadic setting. In fact, the formation of $\mathbb{N}$ defines also the arithmetical operations. Indeed, $\mathbb{N}$ is formed by means of the successor (`succ`) operation, and other arithmetical operations on $\mathbb{N}$ are based on `succ`, that is, they are based on *enumeration*. In this paper we critically deliberate about extentions in the sense of Example 1 and extensions by means of ZEP.

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2 Monads and monad compositions

Monads date back to 1958 and work by Godement, and also to 1961 with work by Huber who showed that adjoint pairs give rise to monads. Lawvere [8] introduced universal algebra into category theory and this can be seen as the the introduction of the term monad. These developments include all required categorical techniques for substitution theories. Let C be a category. A monad (or triple, or algebraic theory) over C is written as $\mathbf{F} = (F, \eta, \mu)$, where $F : \mathsf{C} \to \mathsf{C}$ is a (covariant) functor, and $\eta : id \to F$ and $\mu : F \circ F \to F$ are natural transformations for which $\mu \circ F\mu = \mu \circ \mu F$ and $\mu \circ F\eta = \mu \circ \eta F = id_F$ hold. In the sequel, let L be a completely distributive lattice. The covariant power-set functor L_{id} is obtained by $L_{id}X = L^X$, and for a morphism $X \xrightarrow{f} Y$ in `Set` we have ([7, 11])

$$\begin{aligned} L_{id}f(A)(y) &= \bigvee_{x \in X} A(x) \wedge f^{-1}(\{y\})(x) \\ &= \bigvee_{f(x)=y} A(x). \end{aligned} \tag{1}$$

Further, define $\eta_X : X \to L_{id}X$ by

$$\eta_X(x)(x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and $\mu_X : L_{id}L_{id}X \to L_{id}X$ by

$$\mu_X(\mathcal{A})(x) = \bigvee_{A \in L_{id}X} A(x) \wedge \mathcal{A}(A). \tag{3}$$

We refer [1, 10] for more detailed discussion on power-set functors. Especially, L_{id} is categorically a correct choice to powerset operators in the sense of Rodabaugh ([10]). Moreover, it is clear that ZEP and (1) coincide when $n = 1$. It is well known that the functor L_{id} can be extended to monad with η and μ defined in (2) and (3), respectively. Indeed, the following proposition can be presented:

Proposition 1 ([9])**.** $\mathbf{L}_{id} = (L_{id}, \eta, \mu)$ is a monad.

Note that $\mathbf{2}_{id}$ is the usual covariant power-set monad $\mathbf{P} = (P, \eta, \mu)$, where PX is the set of subsets of X, $\eta_X(x) = \{x\}$ and $\mu_X(\mathcal{B}) = \bigcup \mathcal{B}$, where $\mathcal{B} \in PPX$. The problem of extending a functor to a monad is not a trivial one, and some strange situations may well arise as shown below. The id^2 functor can be extended to a monad with $\eta_X(x) = (x, x)$ and $\mu_X((x_1, x_2), (x_3, x_4)) = (x_1, x_4)$. Similarly, id^n can be extended to a monad. In addition, the proper power-set functor P_0, where $P_0X = PX \setminus \{\emptyset\}$, as well as $id^2 \circ P_0$ can, respectively, be extended to a monad in a unique way. However, $P_0 \circ id^2$ cannot be made to a monad [4].

Remark 1. Let $\mathbf{\Phi} = (\Phi, \eta^\Phi, \mu^\Phi)$ and $\mathbf{\Psi} = (\Psi, \eta^\Psi, \mu^\Psi)$ be monads over `Set`. The composition $\Phi \circ \Psi$ cannot always be extended to a monad as we see in the case of $P_0 \circ id^2$.

Especially, $L_0 \circ id^2$ cannot be extended to monad, where $L_0X = L_{id}X \setminus \{\emptyset\}$. One might now try the functor $id^2 \circ L_0$ to obtain ZEP as an approach to extend binary operations on X to binary operations on $L_{id}X$. However, the following discussion shows some problems. Consider we have a binary operation $f : X \times X \to X$. It is clear that we can think f as a `Set`-morphism. Applying L_{id} we have then $L_{id}f : L_{id}(X \times X) \to L_{id}X$ such that for any $R \in L_{id}(X \times X)$ and $x, y \in X$,

$$L_{id}f(R)(z) = \bigvee_{z=f(x,y)} R(x, y).$$

Unfortunately, $L_{id}f$ is not a generalization of f in the sense that we should have an operation on $L_{id}X$. Indeed, we would like to have an operation $h : (id^2 \circ L_{id})X \to L_{id}X$, but it is clear that this is possible only if we have a natural transformation $\sigma : L_{id} \to id^2 \circ L_{id}$. Now, for $X \times X$ we have

$$\sigma_{X\times X} : L_{id}(X\times X) \to L_{id}(X\times X)\times L_{id}(X\times X).$$

As a conclusion of this discussion we can say that ZEP may be obtained directly by L_{id} for unary arithmetical operations only. Concerning generalizations of terms, see [1], we adopt a more functorial presentation of the set of terms, as opposed to using the conventional inductive definition of terms, where we bind ourselves to certain styles of proofs. Even if a purely functorial presentation might seem complicated, there are advantages when we define corresponding monads, and, further, a functorial presentation simplifies efforts to prove results concerning compositions of monads. For a set A, the constant set functor $A_{\mathtt{Set}}$ is the covariant set functor which assigns sets X to A, and mappings f to the identity map id_A. The sum $\sum_{i\in I}\varphi_i$ of covariant set functors φ_i assigns to each set X the disjoint union $\bigcup_{i\in I}(\{i\}\times\varphi_i X)$, and to each morphism $X \xrightarrow{f} Y$ in $\mathtt{Set}$ the mapping $(i,m)\mapsto (i,\varphi_i f(m))$, where $(i,m)\in(\sum_{i\in I}\varphi_i)X$. Let k be a cardinal number and $(\Omega_n)_{n\leq k}$ be a family of sets. We will write $\Omega_n id^n$ instead of $(\Omega_n)_{\mathtt{Set}}\times id^n$. Note that $\sum_{n\leq k}\Omega_n id^n X$ is the set of all triples $(n,\omega,(x_i)_{i\leq n})$ with $n\leq k$, $\omega\in\Omega_n$ and $(x_i)_{i\leq n}\in X^n$. A disjoint union $\Omega=\bigcup_{n\leq k}\{n\}\times\Omega_n$ is an operator domain, and an Ω-algebra is a pair $(X,(s_{n\omega})_{(n,\omega)\in\Omega})$ where $s_{n\omega}: X^n\to X$ are n-ary operations. The $\sum_{n\leq k}\Omega_n id^n$-morphisms between Ω-algebras are precisely the homomorphisms between the algebras. The term functor can now be defined by transfinite induction. In fact, let $T^0_\Omega = id$ and define

$$T^\alpha_\Omega = (\sum_{n\leq k}\Omega_n id^n)\circ\bigcup_{\beta<\alpha}T^\beta_\Omega$$

for each positive ordinal α. Finally, let

$$T_\Omega = \bigcup_{\alpha<\bar{k}}T^\alpha_\Omega$$

where $\bar{k}$ is the least cardinal greater than k and $\aleph_0$. Clearly, $(n,\omega,(m_i)_{i\leq n})\in T^\alpha_\Omega X$, $\alpha\neq 0$, implies $m_i\in T^{\beta_i}_\Omega X$, $\beta_i<\alpha$. A morphism $X\xrightarrow{f} Y$ in $\mathtt{Set}$ can also be extended to the corresponding Ω-homomorphism $(T_\Omega X,(\sigma_{n\omega})_{(n,\omega)\in\Omega}) \xrightarrow{T_\Omega f} (T_\Omega Y,(\tau_{n\omega})_{(n,\omega)\in\Omega})$, where $T_\Omega f$ is defined to be the Ω-extension of $X\xrightarrow{f} Y\hookrightarrow T_\Omega Y$ associated to $(T_\Omega Y,(\tau_{n\omega})_{(n,\omega)\in\Omega})$. We can now extend T_Ω to a monad. Define $\eta^{T_\Omega}_X(x)=x$. Further, let $\mu^{T_\Omega}_X = id^\star_{T_\Omega X}$ be the Ω-extension of $id_{T_\Omega X}$ with respect to $(T_\Omega X,(\sigma_{n\omega})_{(n,\omega)\in\Omega})$.

Proposition 2 ([9]). $\mathbf{T}_\Omega=(T_\Omega,\eta^{T_\Omega},\mu^{T_\Omega})$ is a monad.

Proposition 3 ([1]). $(L_{id}T_\Omega,\eta^{L_{id}T_\Omega},\mu^{L_{id}T_\Omega})$, denoted $\mathbf{L}_{id}\bullet\mathbf{T}_\Omega$, is a monad.

3 Fuzzification of arithmetic

In [6] terms are described in a general setting in a substitution theory. This means essentially generalizing the underlying signature to involving usage of the composed monad $\mathbf{L}_{id}\bullet\mathbf{T}_\Omega$. An effort to generalize the notion of sentences can be found in [2]. The composed functor $T_\Omega L_{id}$ on the other hand is problematic as we are not able to extend it to a corresponding monad $\mathbf{T}_\Omega\bullet\mathbf{L}_{id}$. The distinction between $L_{id}T_\Omega$ and $T_\Omega L_{id}$ is important e.g. with respect to approaches to fuzzy arithmetic, as we need to understand if 'fuzzy arithmetic' produces terms in $L_{id}T_\Omega$ or $T_\Omega L_{id}$. In the latter a composition of substitutions is not possible as the underlying composed functor is not extendable to a monad. We are thus referred to staying within the set $L_{id}T_\Omega X$, and therefore we are NOT doing 'arithmetic with fuzzy' which has been the default approach for 'fuzzy arithmetic'. Especially, arithmetic need to be defined before fuzzification, thus, Example 1 is not an approach to 'fuzzy arithmetic'. Fuzzy sets of arithmetic expressions, like `approximately` x, are then represented by mappings from $T_\Omega X$ to L. This is in our view intuitively more appealing.

Example 2. [3] Consider the element `approx(x+y)` of LTX, where $L=L_{id}$ and $T=T_\Omega$. With the substitution

```
x := approx 0
y := approx 60
```

applied to `approx(x+y)` we obtain the expression

```
approx(approx 0 + approx 60)
```

which is an element of $LTLTX$. However, applying μ^{LT}_X on `approx(approx 0 + approx 60)` brings μ^{LT}_X(`approx(approx 0 + approx 60)`) to become an element of LTX.

Let us focus on semantics, fuzzy natural numbers again. It is clear that ZEP can be applied to unary operations, which can be seen as follows:

Example 3. Consider $(\mathbb{N}, \mathtt{succ})$, where $\mathtt{succ}\colon \mathbb{N} \to \mathbb{N}$, is the successor operation. It is clear that $\mathtt{succ}$ can be extended to $\mathtt{fsucc} = L_{id}\,\mathtt{succ}$ by means of ZEP. In fact this is just a 'shift' by one unit to the right for $A \in L_{id}\mathbb{N}$. Notice that ZEP determines $\mathtt{fsucc}(A)(0) = 0$.

Let us now define a mapping $\mathtt{succ}^A : \mathbb{N} \to L_{id}\mathbb{N}$ such that for all $m \in \mathbb{N}$

$$\mathtt{succ}^A(m)(n) = \bigvee_{\eta(n)=\mathtt{succ}^{\eta(\mathtt{k})}(\mathtt{m})} A(k)$$

It is clear we may generalize $\mathtt{succ}^A$ to $L_{id}\mathtt{succ}^A\colon L_{id}\mathbb{N} \to L_{id}L_{id}\mathbb{N}$ and the apply $\mu_{\mathbb{N}}$. Thus, we can set for all $B \in \mathbb{N}$,

$$B + A = (\mu_{\mathbb{N}} \circ L_{id}\mathtt{succ}^A)(B) \qquad (4)$$

Proposition 4. Equation (4) coincides with ZEP.

Proof. We have

$$\begin{aligned}
(B+A)(y) &= \mu_{\mathbb{N}}(L_{id}\mathtt{succ}^A(B))(y)\\
&= \bigvee_{D\in L_{id}\mathbb{N}} D(y) \wedge L_{id}\mathtt{succ}^A(B)(D)\\
&= \bigvee_{D\in L_{id}\mathbb{N}} \bigvee_{\mathtt{succ}^A(x)=D} D(y) \wedge B(x)\\
&= \bigvee_{x\in\mathbb{N}} \mathtt{succ}^A(x)(y) \wedge B(x)\\
&= \bigvee_{x\in\mathbb{N}} \left(\bigvee_{y=\mathtt{succ}^k(x)} A(k) \right) \wedge B(x)\\
&= \bigvee_{x\in\mathbb{N}} \bigvee_{y=\mathtt{succ}^k(x)} B(x) \wedge A(k)\\
&= \bigvee_{y=x+k} B(x) \wedge A(k).
\end{aligned}$$

□

Indeed it is clear that Proposition 4 confirms Proposition 1, or at least can be seen as a corollary to Proposition 1, which, of course, was observed already in [9]. The current paper, in fact, takes this observation as a starting point for further work. Whereas Manes ([9]) uses $\mathbf{L}_{id}$ only, we suggest in the future to take further steps and replace the functor L_{id} with suitable other functors extendable to monads. This then provides the instrumentation that enables ZEP to be generalized using monadic techniques to apply far beyond just using $\mathbf{L}_{id}$. Note also that the multiplication of a monad is rarely unique, or at least to say that it is not known whether uniqueness of a multiplication of a monad is a rule rather than an exception. Note also that the multiplication of a monad is rarely unique, or at least to say that it is not known whether uniqueness of a multiplication of a monad is a rule rather than an exception. The uniqueness of the identity, on the other hand, is mostly true ([4]). This immediately then binds ZEP to the (conjectured) non-uniqueness of the multiplication in $\mathbf{L}_{id}$, and more generally, to any multiplication in a monad based generalized view of ZEP.

4 Conclusion

Fuzzy numbers need to be viewed based on their underlying set functors together with the operators found in selected algebras. Extendability to monads is important as we additionally need to handle substitutions, and indeed compositions of substitutions. Examples in this paper clearly motivates to use categorical machinery for fuzzy arithmetics utilizing its underlying signatures and algebras. However, it is not clear which kind of signature and equational logic would have the described semantics. The impact for fuzzy arithmetics is yet to be seen and we will develop our constructions further in future works.

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