

New Algorithms for Trapezoidal Approximation of Fuzzy Numbers Preserving the Expected Interval

Przemysław Grzegorzewski

Systems Research Institute, Polish Academy of Sciences,
Newelska 6, 01-447 Warsaw, Poland
pgrzeg@ibspan.waw.pl

Abstract

Fuzzy number approximation by trapezoidal fuzzy numbers which preserves the expected interval is discussed. New algorithms for calculating the proper approximations are proposed. It is shown that the adequate approximation operator is chosen with respect both to the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number.

Keywords: fuzzy numbers, approximation of fuzzy numbers, expected interval, expected value, width, ambiguity, value of a fuzzy number.

1 Introduction

Trapezoidal approximation of fuzzy numbers was considered by many authors (see, e.g. [1], [2], [5], [11], [12], [13], [15], [16]). In [12] a list of criteria which trapezoidal approximation operators should possess was formulated and a new approach to trapezoidal approximation that lead to, so-called, the nearest trapezoidal approximation operator preserving the expected interval was suggested. Then in [13] a corrected version of that operator was given but the ultimate shape of that approximation operator was presented in [2] and [16]. It appears that the form of the nearest trapezoidal approximation operator preserving the

expected interval depends on the particular shape of a fuzzy number to be approximated. Actually, a given fuzzy number might be approximated by one of the four admissible approximation operators. Which one should be used depends on parameters that characterize the location and spread of a fuzzy number, i.e. on its value, weighted expected value, ambiguity and width, respectively. These conditions together with natural algorithms for computing the nearest trapezoidal approximation preserving the expected interval were given in [11].

One may ask why the preservation of the expected interval is so exposed. There are many reasons to do so starting from the important properties of the expected interval itself (see, e.g. [8], [14]). But there are also some other interesting properties of fuzzy numbers which remain invariant under approximation provided the expected interval remains unchanged (see [12]).

In the present paper we suggest modified conditions for choosing the proper approximation operator. They are both simpler than discussed previously and they have more natural interpretation. One of the suggested algorithms indicates that the adequate approximation operator is chosen with respect both to the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number. Moreover, another mathematical formulae for the operators under discussion are proposed.

2 Concepts and notations

Let A denote a *fuzzy number*, i.e. such fuzzy subset A of the real line \mathbb{R} with membership function $\mu_A : \mathbb{R} \rightarrow [0, 1]$ which is (see [7]): normal (i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$), fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$, $\forall x_1, x_2 \in \mathbb{R}, \forall \lambda \in [0, 1]$), μ_A is upper semicontinuous, $\text{supp}A$ is bounded, where $\text{supp}A = \text{cl}(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and cl is the closure operator. A space of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Moreover, let $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$, $\alpha \in (0, 1]$, denote an α -cut of a fuzzy number A . As it is known, every α -cut of a fuzzy number is a closed interval, i.e. $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$.

The *expected interval* $EI(A)$ of a fuzzy number A is given by (see [8], [14])

$$EI(A) = \left[\int_0^1 A_L(\alpha) d\alpha, \int_0^1 A_U(\alpha) d\alpha \right]. \quad (1)$$

The middle point of the expected interval given by

$$EV(A) = \frac{1}{2} \left(\int_0^1 A_L(\alpha) d\alpha + \int_0^1 A_U(\alpha) d\alpha \right) \quad (2)$$

is called the *expected value* of a fuzzy number and it represents the typical value of the fuzzy number A (see [8], [14]). Sometimes its generalization, called *weighted expected value*, might be interesting. It is defined by

$$EV_q(A) = (1-q) \int_0^1 A_L(\alpha) d\alpha + q \int_0^1 A_U(\alpha) d\alpha, \quad (3)$$

where $q \in [0, 1]$ (see [9]).

Another useful parameter characterizing the nonspecificity of a fuzzy number is called the *width* of a fuzzy number (see [4]) and is defined by

$$\begin{aligned} w(A) &= \int_{-\infty}^{\infty} \mu_A(x) dx \\ &= \int_0^1 (A_U(\alpha) - A_L(\alpha)) d\alpha. \end{aligned} \quad (4)$$

To simplify the representation of fuzzy numbers Delgado et al. [5] suggested two parameters – value and ambiguity – which represent some basic features of fuzzy numbers and hence they were called a canonical representation of fuzzy numbers. The first notion

$$Val(A) = \int_0^1 \alpha (A_L(\alpha) + A_U(\alpha)) d\alpha \quad (5)$$

is called the *value* of fuzzy number A and might be seen as a point that corresponds to the typical value of the magnitude that the fuzzy number A represents. The next index, called the *ambiguity* is given by

$$Amb(A) = \int_0^1 \alpha (A_U(\alpha) - A_L(\alpha)) d\alpha, \quad (6)$$

and it characterizes the global spread of the membership function and hence is a measure of vagueness of fuzzy number A .

For two arbitrary fuzzy numbers A and B with α -cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$\begin{aligned} d(A, B) &= \left(\int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha \right. \\ &\quad \left. + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha \right)^{1/2} \end{aligned} \quad (7)$$

is the distance between A and B (for more details we refer the reader to [9]). There are, of course, some other measures of the distance between fuzzy numbers (e.g. [3]), however (7) is not only very popular but it seems to be especially useful in relation with the expected interval (see [10]).

3 Trapezoidal approximation

Suppose we are looking for an approximation operator T which produces a trapezoidal fuzzy number $T(A)$ that is the closest with respect to distance (7) to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one. More precisely, we want substitute a fuzzy number A by the trapezoidal fuzzy number $T(A)$, i.e. by a fuzzy number

with linear sides and the membership function having a following form:

$$\mu_{T(A)}(x) = \begin{cases} 0 & \text{if } x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if } t_1 \leq x < t_2, \\ 1 & \text{if } t_2 \leq x \leq t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if } t_3 < x \leq t_4, \\ 0 & \text{if } t_4 < x. \end{cases} \quad (8)$$

Moreover, since the desired operator should preserve the expected interval of a fuzzy number, a following condition should be fulfilled

$$EI(T(A)) = EI(A). \quad (9)$$

However, since a trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core, our goal reduces to finding such real numbers $t_1 \leq t_2 \leq t_3 \leq t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$. Such operator is called *the nearest trapezoidal approximation operator preserving the expected interval* (actually, T is d -nearest trapezoidal approximation operator, where D is given by 7), however further on we call it, in brief, just the nearest one).

The solution of the above mentioned problem was suggested firstly in [12] and later it was improved in [13]. Although operators given in these papers generally produce proper approximations, one can construct such fuzzy number that they do not work correctly and the output is not a trapezoidal fuzzy number. Therefore, Ban [2] and Yeh [16] removed this gap and proposed a final solution containing four possible operators $T_i(A) = T_i(t_1, t_2, t_3, t_4)$, $i = 1, \dots, 4$. Then Grzegorzewski [11] simplified the requirements for choosing the proper approximation operators. According to [11] we get:

(a) if

$$Amb(A) \geq \frac{1}{3}w(A) \quad (10)$$

then the solution $T_1(A) = T_1(t_1, t_2, t_3, t_4)$ is

given by

$$t_1 = 4 \int_0^1 A_L(\alpha) d\alpha - 6 \int_0^1 \alpha A_L(\alpha) d\alpha \quad (11)$$

$$t_2 = -2 \int_0^1 A_L(\alpha) d\alpha + 6 \int_0^1 \alpha A_L(\alpha) d\alpha \quad (12)$$

$$t_3 = -2 \int_0^1 A_U(\alpha) d\alpha + 6 \int_0^1 \alpha A_U(\alpha) d\alpha \quad (13)$$

$$t_4 = 4 \int_0^1 A_U(\alpha) d\alpha - 6 \int_0^1 \alpha A_U(\alpha) d\alpha \quad (14)$$

(b) if

$$Amb(A) < \frac{1}{3}w(A) \quad (15)$$

and

$$EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A) \quad (16)$$

then we get $T_2(A) = T_2(t_1, t_2, t_3, t_4)$, where

$$t_1 = 3 \int_0^1 A_L(\alpha) d\alpha + \int_0^1 A_U(\alpha) d\alpha - 3 \int_0^1 \alpha A_L(\alpha) d\alpha - 3 \int_0^1 \alpha A_U(\alpha) d\alpha \quad (17)$$

$$t_2 = - \int_0^1 A_L(\alpha) d\alpha - \int_0^1 A_U(\alpha) d\alpha + 3 \int_0^1 \alpha A_L(\alpha) d\alpha + 3 \int_0^1 \alpha A_U(\alpha) d\alpha \quad (18)$$

$$t_3 = t_2 \quad (19)$$

$$t_4 = \int_0^1 A_L(\alpha) d\alpha + 3 \int_0^1 A_U(\alpha) d\alpha - 3 \int_0^1 \alpha A_L(\alpha) d\alpha - 3 \int_0^1 \alpha A_U(\alpha) d\alpha \quad (20)$$

(c) if

$$Val(A) < EV_{\frac{1}{3}}(A) \quad (21)$$

then we get $T_3(A) = T_3(t_1, t_2, t_3, t_4)$ given by

$$t_1 = t_2 = t_3 = \int_0^1 A_L(\alpha) d\alpha \quad (22)$$

$$t_4 = 2 \int_0^1 A_U(\alpha) d\alpha - \int_0^1 A_L(\alpha) d\alpha \quad (23)$$

(d) if
$$Val(A) > EV_{\frac{2}{3}}(A) \quad (24)$$

then we obtain $T_4(A) = T_4(t_1, t_2, t_3, t_4)$ such that

$$t_1 = 2 \int_0^1 A_L(\alpha) d\alpha - \int_0^1 A_U(\alpha) d\alpha \quad (25)$$

$$t_2 = t_3 = t_4 = \int_0^1 A_U(\alpha) d\alpha. \quad (26)$$

To sum up we get a following algorithm for computing the nearest trapezoidal approximation preserving the expected interval.

Algorithm 1

Step 1. If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (11)-(14), else

Step 2. if $EV_{\frac{1}{3}}(A) \leq Val(A) \leq EV_{\frac{2}{3}}(A)$ then apply operator T_2 given by (17)-(20), else

Step 3. if $Val(A) < EV_{\frac{1}{3}}(A)$ then apply operator T_3 given by (22)-(23), else

Step 4. apply operator T_4 given by (25)-(26).

Therefore, we have received four different operators providing the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number, where T_1 leads to trapezoidal fuzzy number, T_2 stands for the operator that leads to triangular fuzzy number with two sides, while T_3 and T_4 produce triangular fuzzy numbers with the right side only or with the left side only, respectively. We approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided ambiguity of this fuzzy number is greater than one third of its width. Otherwise, we approximate A by a triangular number. It means that for less vague fuzzy numbers the solution is always a triangular fuzzy number. Thus, to sum up, the distinction between possible solutions - either trapezoidal $T_1(A)$ or triangular $T_2(A)$ - depends on the relationship between two parameters of the original fuzzy number that describe its dispersion. In other words, to approximate a fuzzy

number A we apply operator T_2 provided A has only slight ambiguity and its typical value is located neither close to the left nor to the right border of its support. However, a fuzzy number with its value Val located close to the left border of its support would be approximated by a triangular fuzzy number with the right side only, produced by operator T_3 , while a fuzzy number with its value Val located close to the right border of its support would be approximated by a triangular fuzzy number with the left side only, produced by operator T_4 .

4 Discussion and new algorithms

Although the given above explanation of the conditions that delimits situations corresponding to different approximation operators is correct, yet it sounds slightly insufficient. Especially conditions related to the location parameters do not have clear interpretation. However, it appears that we can propose equivalent conditions which seem to be more satisfactory and more natural.

Since by (3)

$$EV_{\frac{2}{3}}(A) = \frac{1}{3} \int_0^1 A_L(\alpha) d\alpha + \frac{2}{3} \int_0^1 A_U(\alpha) d\alpha$$

$$EV_{\frac{1}{3}}(A) = \frac{2}{3} \int_0^1 A_L(\alpha) d\alpha + \frac{1}{3} \int_0^1 A_U(\alpha) d\alpha$$

then according to (2) and (4) we get

$$EV_{\frac{2}{3}}(A) + EV_{\frac{1}{3}}(A) =$$

$$= \int_0^1 A_L(\alpha) d\alpha + \int_0^1 A_U(\alpha) d\alpha$$

$$= 2EV(A)$$

and

$$EV_{\frac{2}{3}}(A) - EV_{\frac{1}{3}}(A) =$$

$$= \frac{1}{3} \int_0^1 A_U(\alpha) d\alpha - \frac{1}{3} \int_0^1 A_L(\alpha) d\alpha$$

$$= \frac{1}{3}w(A).$$

Hence

$$EV_{\frac{2}{3}}(A) = EV(A) + \frac{1}{6}w(A)$$

and

$$EV_{\frac{1}{3}}(A) = EV(A) - \frac{1}{6}w(A).$$

Thus condition (16) is equivalent to

$$|EV(A) - Val(A)| \leq \frac{1}{6}w(A). \quad (27)$$

Similarly, requirement (21) might be replaced by

$$Val(A) < EV(A) - \frac{1}{6}w(A), \quad (28)$$

while (24) is equivalent to

$$Val(A) > EV(A) + \frac{1}{6}w(A). \quad (29)$$

Thus we get another algorithm.

Algorithm 2

Step 1. If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (11)-(14), else

Step 2. if $|EV(A) - Val(A)| \leq \frac{1}{6}w(A)$ then apply operator T_2 given by (17)-(20), else

Step 3. if $Val(A) > EV(A) + \frac{1}{6}w(A)$ then apply operator T_4 given by (25)-(26), else

Step 4. apply operator T_3 given by (22)-(23).

As it is seen now to find a proper approximation using Algorithm 2 we have to calculate at most 4 parameters, while in previous Algorithm 1 we had 5 parameters.

To emphasize much more that the distinction among operators T_2 , T_3 and T_4 is based on the asymmetry of spread of the membership function let us introduce following notions.

Definition 1

The left-hand ambiguity of a fuzzy number A with α -cuts $A_\alpha = [A_L(\alpha), A_U(\alpha)]$ is defined by

$$Amb_L(A) = \int_0^1 \alpha [EV(A) - A_L(\alpha)] d\alpha, \quad (30)$$

while the right-hand ambiguity of a fuzzy number A is given by

$$Amb_U(A) = \int_0^1 \alpha [A_U(\alpha) - EV(A)] d\alpha. \quad (31)$$

One may notice that our definition of the left-hand and right-hand ambiguity differs from the definitions proposed in [6] where the center point of the core of a fuzzy number is placed instead of $EV(A)$. Assuming that the expected value of a fuzzy number $EV(A)$ characterizes its typical value the left-hand and right-hand ambiguity describe the spread of the the left-hand and right-hand part of a fuzzy number, respectively. Moreover, as it is easily seen, both characteristics give the total spread of a fuzzy number, i.e.

$$Amb_L(A) + Amb_U(A) = Amb(A). \quad (32)$$

For our further considerations the following notion would be useful.

Definition 2

The difference between the left-hand and right-hand ambiguity of a fuzzy number A is defined by

$$\Delta Amb(A) = Amb_U(A) - Amb_L(A) \quad (33)$$

By (5) and (2) we get immediately that

$$\Delta Amb(A) = Val(A) - EV(A). \quad (34)$$

Therefore we get another condition equivalent both to (16) and (27), i.e.

$$|\Delta Amb(A)| \leq \frac{1}{6}w(A). \quad (35)$$

Similarly, condition

$$\Delta Amb(A) > \frac{1}{6}w(A) \quad (36)$$

is equivalent to (24) and (29), while condition

$$\Delta Amb(A) < -\frac{1}{6}w(A), \quad (37)$$

i.e.

$$Amb_L(A) - Amb_U(A) > \frac{1}{6}w(A) \quad (38)$$

is equivalent to (21) and (28).

Using these requirements we obtain another algorithm for computing the nearest trapezoidal approximation preserving the expected interval.

Algorithm 3

Step 1. If $Amb(A) \geq \frac{1}{3}w(A)$ then apply operator T_1 given by (11)-(14), else

Step 2. if $|\Delta Amb(A)| \leq \frac{1}{6}w(A)$ then apply operator T_2 given by (17)-(20), else

Step 3. if $\Delta Amb(A) > \frac{1}{6}w(A)$ then apply operator T_4 given by (25)-(26), else

Step 4. apply operator T_3 given by (22)-(23).

It is worth noticing that Algorithm 3 utilizes 3 parameters only and a final decision for the proper choice of the approximation operator depends both on the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of a fuzzy number.

Last of all let us notice that we may express formulae describing operators $T_1 - T_4$ using suitable parameters describing fuzzy numbers instead of relevant integrals given in Section 3. In particular we obtain a very natural formulae especially for operators T_3 and T_4 , namely:

- $T_3(A) = T_3(t_1, t_2, t_3, t_4)$ is given by

$$t_1 = t_2 = t_3 = EV(A) - \frac{1}{2}w(A) \quad (39)$$

$$t_4 = EV(A) + \frac{3}{2}w(A) \quad (40)$$

- $T_4(A) = T_4(t_1, t_2, t_3, t_4)$ is given by

$$t_1 = EV(A) - \frac{3}{2}w(A) \quad (41)$$

$$t_2 = t_3 = t_4 = EV(A) + \frac{1}{2}w(A). \quad (42)$$

After some calculations we also get formulae for operators T_2 and T_1 which are more compact than the original:

- $T_2(A) = T_2(t_1, t_2, t_3, t_4)$, where

$$t_1 = EV(A) - w(A) - 3\Delta Amb(A) \quad (43)$$

$$t_2 = t_3 = EV(A) - \Delta Amb(A) \quad (44)$$

$$t_4 = EV(A) + w(A) - 3\Delta Amb(A) \quad (45)$$

- $T_1(A) = T_1(t_1, t_2, t_3, t_4)$, where

$$t_1 = EV(A) - 2w(A) + 6Amb_L(A) \quad (46)$$

$$t_2 = EV(A) + w(A) - 6Amb_L(A) \quad (47)$$

$$t_3 = EV(A) - w(A) + 6Amb_U(A) \quad (48)$$

$$t_4 = EV(A) + 2w(A) - 6Amb_U(A). \quad (49)$$

As it is seen, in all cases the crucial point of the trapezoidal fuzzy number obtained as the approximation is the expected value the original fuzzy number which is invariant under approximation (see [11]). Then all points $t_1 - t_4$ that describe the trapezoidal fuzzy number are obtained by adding or subtracting some multiplicities of different measures of spread of the original fuzzy number.

5 Conclusions

In the present contribution we have continued the discussion on the problem of trapezoidal approximation of fuzzy numbers showing another algorithms for computing the proper nearest trapezoidal approximation preserving the expected interval. It seems that these new algorithm are simpler and have more clear interpretation than the algorithms proposed before. Especially Algorithm 3 is the most concise and shows that the choice of the adequate approximation operator depends both on the global spread of a fuzzy number and the size of possible asymmetry between the spread of the left-hand and right-hand part of the original fuzzy number.

References

- [1] T. Allahviranloo, M.A. Firozja, *Note on trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems 158 (2007), 755–756.
- [2] Ban A., *Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval*, Fuzzy Sets and Systems (to appear).
- [3] Bertoluzza C., Corral N., Salas A., *On a new class of distances between fuzzy numbers*, Mathware and Soft Computing 2 (1995), 71–84.

- [4] S. Chanas, *On the interval approximation of a fuzzy number*, Fuzzy Sets and Systems 122 (2001), 353–356.
- [5] M. Delgado, M.A. Vila, W. Voxman, *On a canonical representation of a fuzzy number*, Fuzzy Sets and Systems 93 (1998), 125–135.
- [6] M. Delgado, M.A. Vila, W. Voxman, *A fuzziness measure for fuzzy number: applications*, Fuzzy Sets and Systems 94 (1998), 205–216.
- [7] D. Dubois, H. Prade, *Operations on fuzzy numbers*, Int. J. Syst. Sci. 9 (1978), 613–626.
- [8] D. Dubois, H. Prade, *The mean value of a fuzzy number*, Fuzzy Sets and Systems 24 (1987), 279–300.
- [9] P. Grzegorzewski, *Metrics and orders in space of fuzzy numbers*, Fuzzy Sets and Systems 97 (1998), 83–94.
- [10] Grzegorzewski P., *Nearest interval approximation of a fuzzy number*, Fuzzy Sets and Systems 130 (2002), 321–330.
- [11] P. Grzegorzewski, *Trapezoidal approximations of fuzzy numbers preserving the expected interval - algorithms and properties*, Fuzzy Sets and Systems (to appear).
- [12] P. Grzegorzewski, E. Mrówka, *Trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems 153 (2005), 115–135.
- [13] P. Grzegorzewski, E. Mrówka, *Trapezoidal approximations of fuzzy numbers - revisited*, Fuzzy Sets and Systems 158 (2007), 757–768.
- [14] S. Heilpern, *The expected value of a fuzzy number*, Fuzzy Sets and Systems 47 (1992), 81–86.
- [15] C.T. Yeh, *A note on trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems 158 (2007), 747–754.
- [16] C.T. Yeh, *Trapezoidal and triangular approximations preserving the expected interval*, Fuzzy Sets and Systems (to appear).