A COMPARATIVE STUDY OF T-INTERVAL ORDERS, T-FERRERS AND T-BIORDERS

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Abstract

Interval orders play a significant role in preference modeling since they do not impose the transitivity of the indifference relation. For crisp relations this concept can be expressed in different equivalent ways. For fuzzy relations these definitions depend on the t-norm we employ and they are no longer equivalent. In this work we study the connection among the different notions of fuzzy interval order. We prove that depending on the t-norm chosen, some of the definitions are more restrictive than others.

Keywords: Additive preference structure, completeness, interval order, Ferrers property, biorder.

1 Introduction

Preference structures are at the basis of preference modeling theory. Given their importance, a large number of works devoted to the topic can be found in the literature (see for example [14, 15, 16], among many others). They are triplets of binary relations that contain all the information about the preferences of a decision maker over a set of alternatives. The three relations they include are called strict preference, indifference and incomparability relations. They cover the three answers the decision maker can give when comparing two of the alternatives. The preference structures handled in the classical theory involve crisp relations. Therefore, they are sometimes not precise enough for capturing real life.

In order to model human decisions in a more accurate way, fuzzy relations were introduced [19]. They take a value between 0 and 1. For a fuzzy relation Q, that value expresses the strength of the connection by Q between the two alternatives considered.

Since fuzzy preference relations adapt better to real life, they have become a topic of interest (see for example, [1, 3, 4, 6, 7, 9, 10]).

Some preference structures like complete preorders impose both the transitivity of the strict preference and indifference relations. However, a large number of experimental studies reveal that individuals are inconsistent with the transitivity of the indifference (see [8, 17], among others). An interval order is a preference structure that models coherent decisions (the strict preference relation must be transitive), and that is consistent with real world: its indifference relation may be intransitive. The class of interval orders is one of the most important classes of preference structures without incomparability studied in the theory of classical preference modeling. In that case, this concept can be formalized by means of different equivalent compositions of preference (large and strong) and indifference relations. In this work we study fuzzy preference structures and the connection among those (equivalent in the crisp sets context) properties.

The work is organized as follows. In the following section we recall some basic notions concerning crisp preference structures. In Section 3 we recall the notion of additive fuzzy preference structure and some definitions for fuzzy relations that generalize the ones presented in Section 2. Section 4 contains the equivalences that hold for crisp relations. In Section 5 we investigate those equivalences for fuzzy relations. Finally, in Section 6 we address some conclusions.

2 Crisp preference structures

Suppose that a decision maker wants to judge a set of alternatives A. Given two alternatives, she can act in one of the following three ways: (i) she clearly prefers one to the other; (ii) the two alternatives are indifferent to her; (iii) she is unable to compare the two alternatives. Accordingly, three (binary) relations on A can be defined: the *strict preference relation* P, the *indifference relation* I and the *incomparability relation* J.

Recall that for a relation Q on A, its converse or transpose is defined as $Q^t = \{(b,a) \mid (a,b) \in Q\}$, its complement as $Q^c = \{(a,b) \mid (a,b) \notin Q\}$ and its dual as $Q^d = \left(Q^t\right)^c$. One easily verifies that the quadruplet (P, P^t, I, J) establishes a particular partition of A^2.

Definition 2.1. [14] *A preference structure on A is a triplet (P, I, J) of relations on A that satisfy:*

(i) P is irreflexive, I is reflexive and J is irreflexive;

(ii) P is asymmetrical, I and J are symmetrical;

(iii) $P \cap I = \emptyset$, $P \cap J = \emptyset$ and $I \cap J = \emptyset$;

(iv) $P \cup P^t \cup I \cup J = A^2$.

Every preference structure can be identified with a unique reflexive relation called *large preference relation* $R = P \cup I$. This relation leads back to the preference structure in the following way:

$$(P, I, J) = \left(R \cap R^d, R \cap R^t, R^c \cap R^d\right). \quad (1)$$

We say that a relation R is *complete* if aRb or bRa for all a, b in the set of alternatives A. In [13] it was proven that the large preference relation is complete if and only if the associated preference structure does not connect any pair of alternatives (a, b) by the incomparability relation, that is, if $J = \emptyset$.

$$R \text{ complete} \quad \Leftrightarrow \quad J = \emptyset.$$

A relation R satisfies the *Ferrers property* (see among others [13, 14]) if

$$(aRb \quad \wedge \quad cRd) \quad \Rightarrow \quad (aRd \quad \vee \quad cRb),$$

for all a, b, c, d in A.

A relation R is a *biorder* if

$$(aRb \quad \wedge \quad c\not{R}b \quad \wedge \quad cRd) \quad \Rightarrow \quad aRd.$$

Let us denote the composition of two binary relations Q_1 and Q_2 by $Q_1 \circ Q_2$. That is, $a(Q_1 \circ Q_2)b$ if and only if there exists c such that $aQ_1c \wedge cQ_2b$. Then the equivalent compositional definition of biorder is $R \circ R^d \circ R \subseteq R$.

Definition 2.2. *An* interval order *is a preference structure (P, I, J) such that $J = \emptyset$ and*

$$P \circ I \circ P \subseteq P.$$

3 Fuzzy preference structures

3.1 Triangular norms

We recall that the binary operator $T : [0,1] \times [0,1] \to [0,1]$ is a *triangular norm* or *t-norm* for short if it is commutative, associative, monotone and has 1 as neutral element. The most important t-norms are the *Minimum*: $T_{\mathbf{M}}(x,y) = \min(x,y)$, the *Product* $T_{\mathbf{P}}(x,y) = xy$, *Lukasiewicz*: $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$. and the *Drastic product*:

$$T_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y) & \text{, if } \max(x,y) = 1, \\ 0 & \text{, otherwise.} \end{cases}$$

The drastic product is the smallest t-norm and the minimum t-norm is the greatest one. That is, for any t-norm T it holds that $T_{\mathbf{D}} \leq T \leq T_{\mathbf{M}}$.

On the other hand, we say that a value $x \in (0,1)$ is a zero-divisor of a t-norm T if there exists a value $y \in (0,1)$ such that $T(x,y) = 0$.

The minimum and the product are t-norms that do not admit zero-divisors, while $T_{\mathbf{L}}$ and $T_{\mathbf{D}}$ do admit them.
An important family of t-norms that admit zero-divisors are the rotation invariant t-norms [11]. A *rotation invariant t-norm* is a t-norm T that verifies

$$T(x, y) > z \quad \Leftrightarrow \quad T(x, 1 - z) > 1 - y,$$

for any $x, y, z \in [0, 1]$.
These operators satisfy in particular that $T(x, y) > 0 \Leftrightarrow x + y > 1$.

Analogously, a *triangular conorm* or *t-conorm* is a binary relation $S : [0, 1] \times [0, 1] \to [0, 1]$ commutative, associative, monotone and that has 0 as neutral element.

It is easy to prove that t-norms and t-conorms are closely related. For any t-norm T it holds that the relation S defined as follows

$$S(x, y) = 1 - T(1 - x, 1 - y), \forall (x, y) \in [0, 1]^2,$$

is a t-conorm. It is called *dual t-conorm* of T. Conversely, for each t-conorm S the relation T defined as $T(x, y) = 1 - S(1 - x, 1 - y)$ is a t-norm.

A wide study of t-norms and t-conorms can be found in [12].

3.2 Fuzzy binary relations

Crisp relations do not allow degrees of preference or indifference. This becomes a drawback for modeling real life. Fuzzy binary relations take a value in the interval $[0, 1]$ that expresses the strength of the connection between two elements [19].

A fuzzy binary relation is defined as a function Q from the cartesian product $A \times A$ into the interval $[0, 1]$. For every pair $(a, b) \in A \times A$, the value $Q(a, b)$ shows the degree of truth of the fact that aQb in the crisp sense. The complementary, transpose (or converse) and dual of the binary relation Q are defined for any $(a, b) \in A \times A$ as follows:

$$\begin{aligned} Q^c(a, b) &= 1 - Q(a, b), \\ Q^t(a, b) &= Q(b, a), \\ Q^d(a, b) &= 1 - Q(b, a). \end{aligned}$$

In the fuzzy sets context the notion of completeness admits different generalizations. The two most important notions are the following ones. Given a relation Q we say that it is

- *Strongly complete:*
  if $Q(a, b) = 1$ or $Q(b, a) = 1, \forall a, b \in A$.
- *weakly complete:*
  if $Q(a, b) + Q(b, a) \geq 1, \forall a, b \in A$.

3.3 Fuzzy preference structures

The notion of preference structure in the setting of fuzzy relations was a topic of debate for several years (see [1]) until Van de Walle et al. [18] introduced the notion of additive fuzzy preference structure.

Definition 3.1. *An* additive fuzzy preference structure *on the set of alternatives A is a triplet of fuzzy binary relations (P, I, J) satisfying*

- *P is irreflexive, I is reflexive and J is reflexive.*
- *P is $T_{\mathbf{L}}$-asymmetric ($P \cap_{T_{\mathbf{L}}} P^t = \emptyset$), I and J are symmetric.*
- *$P \cap_{T_{\mathbf{L}}} I = P \cap_{T_{\mathbf{L}}} J = I \cap_{T_{\mathbf{L}}} J = \emptyset$.*
- *$(P \cup_{S_{\mathbf{L}}} I)^c = P^t \cup_{S_{\mathbf{L}}} J$.*

Every preference structure has associated a *fuzzy large preference relation* R obtained as $R = P + I$. The value $R(a, b)$ expresses the degree of truth of the assertion "a is at least as good as b". Given that relation R we can build many different AFPS by means of generators [2]. A *generator* i is a symmetric function $i : [0, 1]^2 \to [0, 1]$, such that $T_L \leq i \leq T_M$. Given a reflexive relation R and a generator i, we can obtain an additive fuzzy preference structure as follows:

$$\begin{aligned} P &= R - i(R, R^t)\,, \\ I &= i(R, R^t)\,, \\ J &= I - (R + R^t - 1)\,. \end{aligned}$$

Additive fuzzy preference structures verify the following important property:

$$P(a, b) + P(b, a) + I(a, b) + J(a, b) = 1.$$

For crisp relations (any of the definitions of) completeness of the large preference relation R is equivalent to the condition $J = \emptyset$. In this context the following equivalence holds:

Proposition 3.1. [5] *Let R be a reflexive fuzzy relation R and J the incomparability relation of the preference structure obtained from R by a generator i. It holds that*

$$J = \emptyset \Leftrightarrow \begin{cases} R \text{ is weakly complete} \\ i|_S = T_L \end{cases}$$

where $S = \{(x, y) \in [0, 1]^2 : \exists (a, b) \in A^2$ *with* $R(a, b) = x, R(b, a) = y\}$.

3.4 Generalized definitions

We introduce some appropriate generalizations of the concepts of Ferrers, biorder and interval order for fuzzy relations and additive fuzzy preference structures. We will stay within the usual setting of t-norms.

Definition 3.2. [10] *Given a t-norm T and a t-conorm S, a fuzzy binary relation Q on A is* (T, S)-Ferrers *if the following inequality holds:*

$$T(Q(a, b), Q(c, d)) \leq S(Q(a, d), Q(c, b))$$

for any $a, b, c, d \in A$.

When S is the dual t-conorm of T, the (T, S)-Ferrers property is just called *T-Ferrers*.

Definition 3.3. [9] *Given a fuzzy binary relation Q on A, Q is a* T-biorder *if it holds that*

$$Q \circ_T Q^d \circ_T Q \subseteq Q\,,$$

or, equivalently,

$$T(T(Q(a, b), 1 - Q(c, b)), Q(c, d)) \leq Q(a, d)$$

for any $a, b, c, d \in A$.

Definition 3.4. [3] *An additive fuzzy preference structure without incomparability (P, I) on a set of alternatives A is called a* T-interval order *if it holds that*

$$P \circ_T I \circ_T P \subseteq P\,,$$

or, equivalently,

$$T(T(P(a, b), I(b, c)), P(c, d)) \leq P(a, d)$$

for any $a, b, c, d \in A$.

Clearly, if $T_1 \leq T_2$, every T_2-Ferrers relation is also T_1-Ferrers, every T_2-biorder is also a T_1-biorder and every T_2-interval order is also a T_1-interval order.

4 Crisp interval order

We first consider classical or crisp relations. In this context, it is well-known ([13, 14]) that

$$R \text{ is Ferrers} \quad \Leftrightarrow \quad R \text{ is a biorder.}$$

It is easy to prove that any reflexive relation satisfying the Ferrers property is complete. We have already commented that the relation R is complete if and only if the associated preference relation does not admit incomparability relation (P, I). In this setting it is also well-known (see again [13, 14]) that

$$P \text{ is Ferrers} \quad \Leftrightarrow \quad R \text{ is Ferrers.}$$

More precisely, for interval orders, Montjardet [13] showed the equivalence among the following notions:

Proposition 4.1. [13] *Let (P, I) be an AFPS without incomparability and R its large preference relation. The following conditions are equivalent.*

a) (P, I) is an interval order.

b) P is Ferrers.

c) P is a biorder.

d) R is Ferrers.

e) R is a biorder.

These definitions are equivalent only if we do require the absence of incomparable elements in the AFPS. If we do not require the relation R to be complete, that is, if we do not require $J = \emptyset$, we can still consider the five previous properties and study their relationship. In that general case, we still have the equivalence between the notion of Ferrers and biorder for a binary relation Q, regardless it is reflexive or not.

Proposition 4.2. *Let Q be a crisp binary relation on A, it holds that*

$$Q \textit{ is Ferrers} \quad \Leftrightarrow \quad Q \textit{ is a biorder.}$$

In particular, we have the equivalence between properties b) and c) and between properties d) and e). It is also easy to prove that whenever R is Ferrers (or a biorder) it is complete. Therefore whenever R is Ferrers, P is Ferrers (a biorder) and $P \circ I \circ P \subseteq P$. It also holds that for any preference structure (P, I, J), if P is Ferrers then $P \circ I \circ P \subseteq P$, but the converse implications do not hold. The following proposition summarizes the relationship among the five conditions of 4.1 when we do not impose any completeness condition.

Proposition 4.3. *Let (P, I, J) be a (crisp) preference structure and let R be its large preference relation. Then,*

i) *(P, I, J) is an interval order, R is a biorder and R is Ferrers are equivalent properties.*

ii) *P is a biorder and P is Ferrers are equivalent properties.*

iii) *(P, I, J) is an interval order implies that P is a biorder. In addition to this, P is a biorder implies $P \circ I \circ P \subseteq P$. The converse implications to the ones showed in this item do not hold.*

The above proposition could be outlined as follows:

$$\begin{gathered}(P, I, J) \text{ int. order} \Leftrightarrow R \text{ Ferrers} \Leftrightarrow R \text{ biorder} \\ \Downarrow \not\Uparrow \\ P \text{ Ferrers} \Leftrightarrow P \text{ biorder} \\ \Downarrow \not\Uparrow \\ P \circ I \circ P \subseteq P\end{gathered}$$

5 Fuzzy interval order

Once we have considered the crisp case, we study the fuzzy sets context. Let us recall that for fuzzy relations every one of the five conditions of Proposition 4.1 admits many different generalizations depending on the t-norm we consider. In contrast to the classical case, we begin by those preference structures that do not admit incomparable alternatives, in this case we consider the converse order. Thus, we start by showing the general results for any additive fuzzy preference structure (with or without incomparable elements). Later, we restrict our study for the case of AFPS whose incomparability relation is empty.

5.1 The general case

The equivalence between Ferrers and biorder still holds for fuzzy relations but only for a particular family of t-norms.

Theorem 5.1. [5] *Consider a t-norm T. Then the following statements are equivalent:*

(i) *Any T-Ferrers relation is a T-biorder.*

(ii) *Any T-biorder is a T-Ferrers relation.*

(iii) *T is rotation-invariant:*

$$T(x, y) \le z \Leftrightarrow T(x, 1 - z) \le 1 - y,$$

for all $x, y, z \in [0, 1]$.

In particular, the equivalence holds for any reflexive relation R and for any strict preference relation P.

If we admit incomparable alternatives, for any AFPS there is a unique large preference relation associated. However, we can get many different preference structures from the same R, one for each generator. Thus, it is not indifferent now to start by the AFPS (P, I, J) or by the associated large preference relation R, as it happens in the crisp case. The results we present next concern the generator $i = T_{\mathbf{L}}$, as it has showed an appropriate behavior (see Proposition 3.1).

Theorem 5.2. *Let us consider a reflexive relation R. Let us consider the preference structure (P, I, J) obtained from R by the generator $i = T_{\mathbf{L}}$ and a t-norm T. Then*

$$R \quad T-\text{Ferrers} \quad \Rightarrow \quad P \quad T-\text{Ferrers}$$

if and only if T satisfies that $T(x, y) > 0$ for all $(x, y) \in [0, 1]^2$ such that $x + y > 1$.

The correspondent implication for T-biorder is fulfilled for a more restrictive class of t-norms, as it shows the following theorem.

Theorem 5.3. *Let us consider a reflexive relation R. Let us consider the preference structure (P, I, J) obtained from R by the generator $i = T_{\mathbf{L}}$ and a t-norm T. Then*

$$R \quad T-\text{biorder} \quad \Rightarrow \quad P \quad T-\text{biorder}$$

if and only if T is rotation-invariant.

We now study the relationship between the property $P \circ_T I \circ_T P \subseteq P$ and the notion of P T-biorder.
It is easy to prove that if P is a T-biorder then it also holds that $P \circ_T I \circ_T P \subseteq P$.

Proposition 5.4. *Let (P, I, J) be an additive fuzzy preference structure and let T be a t-norm. It holds that*

$$P \circ_T P^d \circ_T P \subseteq P \quad \Rightarrow \quad P \circ_T I \circ_T P \subseteq P.$$

This result follows from the additive property that any AFPS satisfies. Given the results obtained for crisp relations, the converse implication does not hold for any t-norm.

From Theorems 5.2, 5.3 and Proposition 5.4 we obtain the following corollaries, where we consider two different starting points: an additive fuzzy preference structure and a reflexive binary relation.

Corollary 5.5. *Let (P, I, J) be an additive fuzzy preference structure and let T be a rotation-invariant t-norm. It holds that P is T-Ferrers if and only if P is a T-biorder and this implies that $P \circ_T I \circ_T P \subseteq P$. The converse implication is not true.*

Corollary 5.6. *Let us consider a reflexive relation R. Let us consider the preference structure (P, I, J) obtained from R by the generator $i = T_{\mathbf{L}}$ and a rotation-invariant t-norm T. Then*

i) R is a T-biorder iff R is T-Ferrers.

ii) P is a T-biorder iff P is T-Ferrers.

iii) If R is a T-biorder, then (P, I, J) is a T-interval order and P is a T-biorder.

5.2 The case $J = \emptyset$

Next we consider a particular case: there are not incomparable elements in the AFPS. Of course, all the previous results presented in the general case can be considered now. However, we can also improve the obtained results or add new ones, when the condition $J = \emptyset$ is imposed, that is, when we consider the decision maker is capable to compare all the outcomes.

Concerning the Ferrers property, it is propagated between the strict preference relation of a preference structure and the associated large preference relation as follows.

Theorem 5.7. [5] *Consider a reflexive binary fuzzy relation R with corresponding AFPS (P, I, J) generated by means of $i = T_{\mathbf{L}}$. The following equivalence holds, for any t-norm T:*

R is weakly complete and T-Ferrers $\Leftrightarrow$ $J = \emptyset$ and P is T-Ferrers.

Concerning the notion of T-biorder, not every t-norm propagates this property.

Theorem 5.8. [5] *Consider a t-norm T. The following statements are equivalent:*

(i) *For any reflexive R with corresponding AFPS (P, I, J) generated by means of $i = T_{\mathbf{L}}$ it holds that if R is a T-biorder, then $J = \emptyset$ and P is a T-biorder.*

(ii) *For any reflexive R with corresponding AFPS (P, I, J) generated by means of $i = T_{\mathbf{L}}$ it holds that if $J = \emptyset$ and P is a T-biorder, then R is a T-biorder.*

(iii) *T is rotation-invariant.*

As we showed in the general case, for any preference structure, if P is a T-biorder, it holds that $P \circ_T I \circ_T P \subseteq P$. The converse implication is not that easy to deal with even for preference structures without incomparability. Although it is false in general, we have found some classes of t-norms for which the implication holds.

Proposition 5.9. *For all t-norm T without zero-divisors,*

$$P \circ_T I \circ_T P \subseteq P \quad \Rightarrow \quad P \quad T-\textit{biorder}$$

Rotation invariant t-norms have a good behaviour when propagating the notion of Ferrers property and biorder. It is therefore reasonable to assume that they will have a good behavior for interval orders. However, we can provide the following proposition.

Proposition 5.10. *Let us consider a t-norm such that it fulfils one of the following requirements:*

1. 0.5 *is a zero divisor of* T.
2. *For a pair* $(x, y) \in (0, 0.5)^2$ *such that* $T(x, y) = 0$*, it holds that* $\min(x, y) \geq (1 - \max(x, y))/2$.

Then there exists a T*-interval order* (P, I) *such that* P *is not a* T*-biorder.*

An immediate consequence of the previous proposition is that if T-interval order implies T-biorder, then $T(x, 1 - x) > 0$ for all $x \in (0, 1)$. Therefore, Proposition 5.9 cannot be extended to rotation-invariant t-norms.

We have also proven that if $P \circ_T I \circ_T P \subseteq P$ implies that P is a T-biorder, then either T has no zero-divisors or all its zero-divisors are smaller than 0.5.

At this point we could think that the implication does not hold for any t-norm with zero divisors. However, we have proven that for some t-norms with zero divisors the implication holds.

Proposition 5.11. *Let* $(T_\alpha)_{\alpha \in (0,1/3)}$ *be the class of t-norms defined as follows*

$$T_\alpha(x, y) = \begin{cases} \min(x, y) & \text{, if } \max(x, y) > \alpha, \\ 0 & \text{, otherwise.} \end{cases}$$

For any preference structure without incomparability (P, I) *such that* $P \circ_{T_\alpha} I \circ_{T_\alpha} P \subseteq P$ *it holds that the preference relation* P *is a* T_α*-biorder.*

From all the previous results, we can conclude the following corollary.

Corollary 5.12. *Let* (P, I) *be an additive fuzzy preference structure without incomparability and let* R *be its large preference relation. Let* T *be a rotation-invariant t-norm. Then the following statement are equivalent:*

i) P *is* T*-Ferrers.*

ii) P *is a* T*-biorder.*

iii) R *is* T*-Ferrers.*

iv) R *is a* T*-biorder.*

Moreover, any of them implies that (P, I) *is* T*-interval order, but the converse is not true in general.*

6 Conclusion

For crisp preference structures whose incomparability relation is empty, there are five different equivalent ways of expressing the concept of interval order. We study if that equivalences hold for fuzzy relations both when the incomparability relation is empty and when we do not impose any completeness condition. We focus on the notion of interval order, since it is one of the most commonly used preference structures. From the results obtained, it follows that this definition is the weakest property of the original five for AFPS without incomparability, when we consider the important class of the rotation-invariant t-norms.

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