Central points and approximation in residuated lattices

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Abstract

Given a subset B of a complete residuated lattice, what are its points which are reasonably close to any point of B? What are the best such points? In this paper, we seek to answer these questions provided closeness is assessed by means of biresiduum, i.e. the truth function of equivalence in fuzzy logic. In addition, we present two algorithms which output, for a given input set M of points in a residuated lattice, another set K which approximates M to a desired degree. We prove that the algorithms are optimal in that the set K is minimal in terms of the number of its elements. Moreover, we show that the elements of any set K' with such property are bounded from below and from above by the elements produced by the two algorithms.

Keywords: Fuzzy Logic, Approximation, Residuated Lattice.

1 Motivation and preliminaries

Suppose there is a collection of metal poles of different lengths with the longest pole having (normalized) length 1. Suppose a person sees a picture of two poles from that collection and is asked to assess their similarity, i.e. the person is asked to tell a degree $p_1 \approx p_2$ to which the poles are similar. $p_1 \approx p_2 = 0$ and $p_1 \approx p_2 = 1$ indicate that the poles are not similar at all and that the poles are indistinguishable, respectively. Since the poles are narrow, the person assesses their similarity based solely on their lengths. The picture does not show a scale, i.e. the person does not know the actual lengths of the poles. An obvious way to asses the similarity s of poles p_1 and p_2 of lengths $l(p_1)$ and $l(p_2)$ is to put

$$p_1 \approx p_2 = \min\left(\frac{l(p_1)}{l(p_2)}, \frac{l(p_2)}{l(p_1)}\right), \qquad (1)$$

i.e. to make the similarity judgment based on the ratio of the lengths. Namely, the ratio does not depend on the actual lengths, i.e. $p_1 \approx p_2 = \min\left(\frac{c\cdot l(p_1)}{c\cdot l(p_2)}, \frac{c\cdot l(p_2)}{c\cdot l(p_1)}\right)$ for any $c > 0$, so it can be assessed even when the person does not know the actual magnification coefficient $c > 0$, i.e. does not know the scale for the picture. Given poles p_1 and p_2 with lengths $l(p_1)$ and $l(p_2)$, what is the length of the pole in the middle? That is, what is the length of the "central pole" p for which

$$p \approx p_1 = p \approx p_2,$$

i.e. for which the similarity to p_1 equals the similarity to p_2? An easy verification shows that the central pole p has length

$$l(p) = \sqrt{l(p_1)} \cdot \sqrt{l(p_2)}. \qquad (2)$$

Suppose now the person knows the scale, i.e. knows the lengths $l(p_1)$ and $l(p_2)$. Then there is another, perhaps more natural, way to assess the similarity. Namely, one can put

$$p_1 \approx p_2 = 1 - |l(p_1) - l(p_2)|, \qquad (3)$$

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i.e. the similarity is proportional to the distance of the normalized lengths of p_1 and p_2. If such measure of similarity is used, the length of the central pole p is

$$l(p) = \frac{l(p_1) + l(p_2)}{2}. \quad (4)$$

Obviously, given a set $B = \{p_1, \ldots, p_n\}$ of poles, the length of the optimal central pole for B is

$$l(p) = \sqrt{\min_i l(p_i)} \cdot \sqrt{\max_i l(p_i)}$$

for similarity given by (1) and

$$l(p) = \frac{\min_i l(p_i) + \max_i l(p_i)}{2}.$$

for similarity given by (3).

In this paper, we present theorems and algorithms motivated by the above types of problems. The first hint to a general framework for this kind of problems is the observation that in (1),

$$p_1 \approx p_2 = l(p_1) \leftrightarrow l(p_2) \quad (5)$$

with $\leftrightarrow$ being the biresiduum corresponding to product t-norm and that in (2),

$$l(p) = m \otimes \sqrt{l(p_1) \leftrightarrow l(p_2)} \quad (6)$$

with $m = \min\{l(p_1), l(p_2)\}$, $\otimes$ denoting the product t-norm and $\sqrt{}$ denoting its square root [4]. Likewise, (5) and (6) become (3) and (4) if $\leftrightarrow$ and $\otimes$ denote the Łukasiewicz biresiduum and t-norm. Henceforth, we consider the framework of left-continuous t-norms and their residua. In fact, we consider a more general framework of complete residuated lattices [6]. Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy so-called adjointness condition, i.e. $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$. Residuated lattices are the main structures of truth degrees used in fuzzy logic [2, 3]. We assume familiarity with examples and basic properties of residuated lattices.

Furthermore, we assume familiarity with basic concepts from tolerance and equivalence relations. Recall that a tolerance relation T on a set U is a reflexive and symmetric relation on U. An equivalence on U is a tolerance which is, moreover, transitive. A block of a tolerance T is a subset B of U for which $B \times B \subseteq T$, i.e. $u\,T\,v$ for every $u, v \in B$. A maximal block of T is a block B of T which is maximal with respect to set inclusion, i.e. such that if $B \subset B'$ then $B' \times B' \not\subseteq T$. A collection of maximal tolerance blocks of T is denoted by U/T. U/T forms a covering of U, i.e. every maximal block is nonempty and the union of all blocks yields U. A class of a tolerance T given by $u \in U$ is the set $[u]_T = \{v \in U \mid u\,T\,v\}$. If T is an equivalence relation, then maximal blocks of T as well as classes of T are just equivalence classes of T.

Given a complete residuated lattice $\mathbf{L}$, denote by $\approx_e$ the tolerance on L defined by

$$a \approx_e b \quad \text{iff} \quad a \leftrightarrow b \geq e$$

and put

$$\begin{aligned} a_e &= e \otimes a, \\ a^e &= e \rightarrow a, \\ [a]_e &= [a_e, (a_e)^e]. \end{aligned}$$

Note that $[p, q]$ denotes the interval $\{x \in L \mid p \leq x \leq q\} \subseteq L$. It can be easily verified that $\approx_e$ a compatible tolerance relation on the complete lattice $\langle L, \leq\rangle$. As a result, the following theorem follows directly from [7]:

Theorem 1 *The factor set $L/{\approx_e}$ is equal to the set $\{[a]_e \mid a \in L\}$.*

2 Maximal blocks and central sets

Let $B \subseteq L$ be a set. We set

$$\begin{aligned} &C_e(B) = \\ &\quad = \{c \in L \mid \text{for } b \in B,\ c \leftrightarrow b \geq e\}. \quad (7) \end{aligned}$$

$C_e(B)$ is called the *e-central set of B* (or simply a central set of B), its elements are called *e-central points of B* (or simply central points of B).

Lemma 1 *$c \in C_e(B)$ iff $(c \rightarrow \bigwedge B) \wedge (\bigvee B \rightarrow c) \geq e$.*

Proof. Follows easily from $c \to (\bigwedge_{b\in B} b) = \bigwedge_{b\in B}(c \to b)$ and $(\bigvee_{b\in B} b) \to c = \bigwedge_{b\in B}(b \to c)$. □

The following theorem shows how to compute the central set $C_e(B)$ of a subset $B \subseteq L$.

Theorem 2 *For any $B \subseteq L$, $C_e(B)$ is equal to $[e \otimes \bigvee B, e \to \bigwedge B]$.*

Proof. By adjointness, $e \leq c \to \bigwedge B$ is equivalent to $c \leq e \to \bigwedge B$ and $e \leq \bigvee B \to c$ is equivalent to $e \otimes \bigvee B \leq c$. Thus the assertion follows from Lemma 1. □

For $c \in L$ set

$$B_e(c) = \{b \in L \mid c \leftrightarrow b \geq e\}. \tag{8}$$

$B_e(c)$ is called the *closed ball with center c and radius e*. Since $c \in B_e(c)$, $B_e(c)$ is always nonempty. A closed ball $B_e(c)$ is called *maximal* if there is no $\bar{c} \neq c$ such that $B_e(c) \subset B_e(\bar{c})$.

Note that a closed ball $B_e(c)$ is exactly the class of tolerance $\approx_e$ determined by c.

Example 1 In the Łukasiewicz structure, $B_e(c)$ is just the interval $[c-(1-e), c+(1-e)] \cap [0,1]$. Hence the closed ball $B_{0.5}(0) = [0, 0.5]$ is not maximal: $B_{0.5}(0) \subset B_{0.5}(0.5) = [0,1]$.

The following result is a simple consequence of the above definitions. Note, however, that it does not say that the central set $C_e(B)$ is not empty.

Lemma 2 *For any subset $B \subseteq L$ and $c \in C_e(B)$ it holds $B \subseteq B_e(c)$.*

Proof. Let $b \in B$. Using (7), we get $c \leftrightarrow b \geq e$, and from (8), we get $b \in B_e(c)$. □

The following theorem provides an easy way to compute any closed ball with given center and radius.

Theorem 3 *For any $c \in L$, the closed ball $B_e(c)$ is equal to the interval $[e \otimes c, e \to c]$.*

Proof. The condition $b \leftrightarrow c \geq e$ from the definition of closed ball has two parts: $b \to c \geq e$ and $c \to b \geq e$. By adjointness, the first part is equivalent to $b \leq e \to c$, the second to $b \geq e \otimes c$. □

Corollary 1 *For any $c \in L$, $c \in C_e(B_e(c))$.*

Proof. From Theorem 2 and Theorem 3 we obtain $C_e(B_e(c)) = [e \otimes (e \to c), e \to (e \otimes c)]$ and from adjointness, $e \otimes (e \to c) \leq c \leq e \to (e \otimes c)$. □

Now we turn our attention to the relationship between closed balls with radius e and blocks of the tolerance $\approx_{e^2}$ (where $e^2 = e \otimes e$) and show that maximal closed balls with radius e coincide with maximal blocks of this tolerance.

Lemma 3 *For any $c \in L$, $B_e(c)$ is a block of $\approx_{e^2}$.*

Proof. From Theorem 3, $e \otimes c$ and $e \to c$ are the least and the greatest elements of $B_e(c)$, respectively. From Theorem 1, the element $e \otimes e \otimes (e \to c)$ is the least element of a maximal block of $\approx_{e^2}$ containing $e \to c$. Since $e \otimes e \otimes (e \to c) \leq e \otimes c$, $B_e(c)$ is contained in this maximal block, which proves the lemma. □

Lemma 4 *For any subset $B \subseteq L$, the central set $C_e(B)$ is nonempty if and only if B is a block of the tolerance $\approx_{e^2}$.*

Proof. According to Theorem 2, the nonemptiness of $C_e(B)$ is equivalent to the condition $e \otimes \bigvee B \leq e \to \bigwedge B$, which is, according to adjointness, equivalent to $e \otimes e \leq \bigvee B \to \bigwedge B$ which is equivalent to the fact that B is a block of $\approx_{e^2}$. □

Now we put together results of the previous lemmas.

Theorem 4 *Every maximal closed ball $B_e(c)$ is a maximal block of $\approx_{e^2}$. Conversely, if $B \subseteq L$ is a maximal block of $\approx_{e^2}$ then the central set $C_e(B)$ is nonempty and for any $c \in C_e(B)$ the closed ball $B_e(c)$ is maximal and equal to B.*

Proof. According to Lemma 3, a maximal ball $B_e(c)$ is a block of $\approx_{e^2}$. It is therefore contained in a maximal block B. From Lemma 4 it follows that this maximal block has at least one central point $c' \in B$. Now, Lemma 2 says that the closed ball $B_e(c')$ contains B. Put together, these considerations give $B_e(c) \subseteq B \subseteq B_e(c')$ and from the maximality of $B_e(c)$ we obtain $B_e(c) = B$.

To prove the converse, we first use Lemma 4 again to obtain $C_e(B) \neq \emptyset$. Now, any closed ball containing $B_e(c)$ should be equal to B, because it is a block of $\approx_{e^2}$ itself (Lemma 3). Any other closed ball containing $B_e(c)$ should be also equal to it by the same reason (Lemma 3). □

3 Optimal central points

So far, we investigated maximal sets which have nonempty sets of e-central points. Now we turn to another problem: find a maximal e such that the set of e-central points of a given set is nonempty.

We say that e is an *admissible radius* of set B if $C_e(B) \neq \emptyset$. From Lemma 2 it follows that if e is an admissible radius of B, then $B \subseteq B_e(c)$ for any $c \in C_e(B)$. Lemma 1 says that for any such c,

$$e \leq (c \to \bigwedge B) \wedge (\bigvee B \to c). \tag{9}$$

An *optimal central point* for $B \subseteq L$ is an element $c \in L$ such that for every m:

$$\bigwedge_{z \in B}(z \leftrightarrow m) \leq \bigwedge_{z \in B}(z \leftrightarrow c).$$

Since for any m we have $\bigwedge_{z \in B}(z \leftrightarrow m) = (m \to \bigwedge B) \wedge (\bigvee B \to m)$ (see, for example, the proof of Lemma 1), c is an optimal central point iff for every m:

$$(m \to \bigwedge B) \wedge (\bigvee B \to m) \leq (c \to \bigwedge B) \wedge (\bigvee B \to c) \tag{10}$$

Theorem 5 1. *For any optimal central point c of B, $e = (c \to \bigwedge B) \wedge (\bigvee B \to c)$ is the largest admissible radius of B.*

2. *If e is the largest admissible radius of B then the set of optimal central points of B is nonempty and is equal to $C_e(B)$.*

Proof. 1. By Lemma 1, $c \in C_e(B)$, which also means that e is an admissible radius of B. Now the assertion follows from (9) and (10).

2. The fact that $C_e(B)$ is nonempty follows directly from definition of admissible radius. Now, for any m, $(m \to \bigwedge B) \wedge (\bigvee B \to m)$ is an admissible radius, hence it is less than or equal to e. On the other hand, for any $c \in C_e(B)$ we have $B \subseteq B_e(c)$, which means $(c \to \bigwedge B) \wedge (\bigvee B \to c) \geq e$. Put together, (10) is satisfied for any $m \in L$, $c \in C_e(B)$. □

Lemma 5 *Let $d = \bigwedge B \to \bigvee B$. Then*

1. *e is an admissible radius of B iff $e \otimes e \leq d$.*

2. *For any $z \in L$, $e = z \wedge (z \to d)$ is an admissible radius of B.*

3. *e is an admissible radius of B iff $e = e \wedge (e \to d)$.*

Proof. 1. $e \otimes e \leq d$ if and only if B is a block of $\approx_{e^2}$, which is equivalent to the requirement that B is a subset of some closed ball $B_e(c)$ (Theorem 4). Hence, c is an e-central point of B.

2. We have $e \otimes e = (z \wedge (z \to d)) \otimes (z \wedge (z \to d)) \leq z \otimes (z \to d) \leq d$ and the result follows from 1.

3. From 1. and adjointness we obtain that e is an admissible radius of B iff $e \leq e \to d$, which is equivalent to $e = e \wedge (e \to d)$. □

Corollary 2 *The set*

$$\{z \wedge (z \to (\bigwedge B \to \bigvee B)) \mid z \in L\} \tag{11}$$

is the set of all admissible radii of B.

Proof. Follows from Lemma 5, parts 2. and 3. □

Theorem 6 (optimal central points) *Set B has optimal central points if and only if the set P from Corollary 2 has a largest element. This element is equal to the corresponding largest admissible radius e.*

Proof. Follows directly from Corollary 2 and Theorem 5. □

Now we derive a simple consequence of the previous results for the case of residuated lattices with square roots. We will use the concept of a square root introduced by Höhle [4]. A complete residuated lattice $\mathbf{L}$ has square roots if there is a function $\sqrt{\ }: L \to L$ satisfying

$$\sqrt{a} \otimes \sqrt{a} = a, \quad (12)$$
$$b \otimes b \leq a \quad \text{implies} \quad b \leq \sqrt{a}, \quad (13)$$

for every $a, b \in L$.

Remark 1 Łukasiewicz, product, and Gödel algebras on $[0, 1]$ have sqaure roots. They are given by

$$\sqrt{a} = \frac{a+1}{2} \quad \text{for Łukasiewicz,}$$
$$\sqrt{a} = \text{ordinary number-theoretic square root of } a \text{ for product,}$$
$$\sqrt{a} = a \quad \text{for Gödel.}$$

Theorem 7 *If $\mathbf{L}$ has square roots then any subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius e it holds*

$$e = \sqrt{\bigwedge B \to \bigvee B}. \quad (14)$$

Proof. According to Lemma 5, part 1. and (12), e is the largest admissible radius of B. The rest follows from Theorem 5, part 1. $\square$

4 Optimal algorithms for approximating sets of truth degrees

We now consider the following type of problems. Given a set M of truth degrees, find a reasonably small set K of truth degrees which approximates M well. We provide a precise statement below. Due to limited scope, we omit proofs in this section.

Note first that such problem naturally arises in the following scenario: Let $A : U \to L$ be a fuzzy set in universe U with $M = \{A(u) \mid u \in U\}$ being the set of truth degrees "used by A". Find a fuzzy set $B : U \to L$ which approximates A well and for which, in addition, the set $K = \{B(u) \mid u \in U\}$ of truth degrees "used by B" is small. In general terms, the advantage of B over A is its simplicity. As an example, B is easier to interpret. Due to the well-known Miller's 7 ± 2 phenomenon [5], people have difficulty to assign and interpret consistently more than 7 ± 2 values of a given variable. So if A represents degrees to which objects (such as products) meet certain criteria, it might be better to present B as an output instead of A.

We consider the following general definition.

A degree $\text{appr}(M, K)$ *to which* $M \subseteq L$ *is approximated by* $K \subseteq L$ is defined by

$$\text{appr}(M, K) = \textstyle\bigwedge_{a \in M} \bigvee_{b \in K} (a \leftrightarrow b). \quad (15)$$

$\text{appr}(M, K)$ can be seen as a truth degree of "for every $a \in M$ there is $b \in K$ such that a and b are similar (close)". Hence, $\text{appr}(M, K)$ can be understood as a natural degree of approximation. Among the basic properties of $\text{appr}(\cdot, \cdot)$ are

1. $\text{appr}(M, K) = 1$ for $M \subseteq K$,
2. $K_1 \subseteq K_2$ implies $\text{appr}(M, K_1) \leq \text{appr}(M, K_2)$.

We now present two problems.

Problem 1 Given (finite) $M \subseteq L$ and a threshold $e \in L$, find (finite) K such that

1. K approximates M to degree at least e, i.e.
$$\text{appr}(M, K) \geq e, \quad (16)$$
2. there is no K' with $|K'| \leq |K|$ for which $\text{appr}(M, K') > e$, i.e. K is a least set in terms of the number of its elements which satisfies (16).

Problem 2 Given (finite) $M \subseteq L$ and a threshold $e \in L$, find (finite) K satisfying 1. and 2. of Problem 1, and

3. For any K' with $|K'| = |K|$,
$$\text{appr}(M, K) \geq \text{appr}(M, K'), \quad (17)$$
i.e. among the sets with $|K|$ elements, K provides the best approximation of M.

In the rest of this section, we assume that the complete residuated lattice $\mathbf{L}$ is linearly ordered, i.e. $a \leq b$ or $b \leq a$ for every $a, b \in L$. The following theorem provides a universal description of sets K satisfying (16).

Theorem 8 *Let $\mathbf{L}$ be linearly ordered.* 1. *Let $\Omega \subseteq 2^L$ be a covering of M and $\varphi\colon \Omega \to L$ a mapping such that for any $B \in \Omega$, $\varphi(B) \in C_e(B)$. Then Ω consists of blocks of the tolerance $\approx_{e^2}$ and for $K = \varphi(\Omega)$, (16) is satisfied.*

2. *If $K \subseteq L$ satisfies (16) then $\Omega = \{B_e(c) \mid c \in K\} \subseteq 2^L$ is a set of blocks of the tolerance $\approx_{e^2}$ which forms a covering of M. Moreover $\varphi\colon \Omega \to L$ defined by $\varphi(B_e(c)) = c$ satisfies $\varphi(B) \in C_e(B)$.*

Proof. Omitted due to lack of space. □

According to Theorem 2, $C_e(B)$ is equal to $[e \otimes \bigvee B, e \to \bigwedge B]$. Hence, we can construct a mapping φ from the first part of the above theorem by setting $\varphi(B)$ to any element of this interval (which is nonempty according to Lemma 4). Obviously, in order for K to provide a good approximation degree $\mathrm{appr}(M, K)$, the best choice is to let $\varphi(B)$ be an optimal central point of B.

Example 2 Let $L = [0,1]^2$ with Łukasiewicz structure, $M = L$, $e = \langle 0.25, 0.25\rangle$. Then

$$K = \{\langle 0,0\rangle, \langle 0.5,0\rangle, \langle 1,0\rangle, \langle 0,0.5\rangle, \langle 0,1\rangle\}$$

satisfies (16). However, for $a = \langle 1,1\rangle$, we have $\bigvee_{b\in K} a \leftrightarrow b = \langle 1,1\rangle$, but $B_e(a) \cap K = \emptyset$ for there is no $b \in K$ such that $a \leftrightarrow b \geq e$. Therefore, assertion 2. from Theorem 8 does not hold.

We now present two algorithms which provide solutions to Problem 1. The first algorithm constructs K by "going up" in the set L of truth degrees.

Algorithm 1

1: `INPUT:` M, e
2: `OUTPUT:` K `satisfying 1. and 2. of Problem 1`
3: $K \leftarrow \emptyset$
4: `while` M `is not empty do`
5: $\quad$ $\mathtt{min} \leftarrow \min(M)$
6: $\quad$ `add` $e \to \mathtt{min}$ `to` K
7: $\quad$ `remove from` M `every element` $\leq (e \otimes e) \to \mathtt{min}$
8: `endwhile`
9: `return` K

The second algorithm constructs K by "going down".

Algorithm 2

1: `INPUT:` M, e
2: `OUTPUT:` K `satisfying 1. and 2. of Problem 1`
3: $K \leftarrow \emptyset$
4: `while` M `is not empty do`
5: $\quad$ $\mathtt{max} \leftarrow \max(M)$
6: $\quad$ `add` $e \otimes \mathtt{max}$ `to` K
7: $\quad$ `remove from` M `every element` $\geq e \otimes e \otimes \mathtt{max}$
8: `endwhile`
9: `return` K

As the next theorem shows, the algorithms stop and are correct, i.e. they produce a set K of minimal size for which $\mathrm{appr}(M, K) \geq e$.

Theorem 9 (termination, correctness) 1. *Algorithms 1 and 2 terminate after at most $|M|$ steps.* 2. *Algorithms 1 and 2 are correct.*

Proof. Omitted due to lack of space. □

Note that Algorithms 1 and 2 work conceptually even for infinite sets M when replacing $\inf(M)$ by $\min(M)$ and $\sup(M)$ by $\max(M)$ in line 5.

Futhermore, the algorithms provide upper and lower bound for every set K' with the minimal number of elements which approximate M to degree at least e.

Theorem 10 (bounds) *Let the sets K^u and K^l produced by Algorithm 1 and Algorithm 2 consist of elements $k_1^u < \cdots < k_m^u$ and $k_1^l < \cdots < k_m^l$, respectively. If K' consisting of $k'_1 < \cdots < k'_m$ satisfies $\mathrm{appr}(M, K') \geq e$, then*

$$k_1^l \leq k'_1 \leq k_1^u, \ldots, k_m^l \leq k'_m \leq k_m^u.$$

Proof. Omitted due to lack of space. □

The following example shows that not every $K' = \{k'_1, \ldots, k'_m\}$ for which $k^l_i \le k'_i \le k^u_i$ satisfies $\mathrm{appr}(M, K') \ge e$.

Example 3 Consider standard Łukasiewicz structure on $L = [0,1]$, $M = \{0.5, 0.7, 0.8\}$, and $e = 0.9$. Then $K^l = \{0.4, 0.7\}$ and $K^u = \{0.6, 0.9\}$. Let $K' = \{0.4, 0.9\}$. Then $\mathrm{appr}(M, K') = 0.8 < 0.9 = e$.

Although the set K produced by Algorithm 1 or Algorithm 2 is optimal in that it is one of the smallest sets for which $\mathrm{appr}(M,K) \ge e$, there can be a set K' of the same size, i.e. $|K'| = |K|$, for which $\mathrm{appr}(M,K') > \mathrm{appr}(M,K)$, i.e. K' provides a better approximation of M than K. From this point of view, the output set K from Algorithm 1 and Algorithm 2 can be improved. Namely, it is easily seen from the description of Algorithm 1 and Algorithm 2 that the set

$$\{B_e(k) \cap M \mid k \in K\}$$

forms a partition of M, i.e. sets $B_e(k) \cap M$ for $k \in K$ are pairwise disjoint and their union is M. Now, in general, k is not an optimal central point of $B_e(k) \cap M$. Therefore, we can improve K by replacing every $k \in K$ by an optimal central point of $B_e(k) \cap M$ (or a point which provides a better apprximation of $B_e(k) \cap M$ than k). By Theorem 7, if $\mathbf{L}$ has square roots, then any element from

$$\Big[\sqrt{\bigwedge(B_e(k) \cap M)} \otimes \bigvee(B_e(k) \cap M), \\ \sqrt{\bigwedge(B_e(k) \cap M)} \to \bigwedge(B_e(k) \cap M)\Big]$$

can be used to replace k. For instance, for $M = \{0.5, 0.7, 0.8\}$ and $K = K^l = \{0.4, 0.7\}$ from Example 3, $B_{0.9}(0.7) \cap M = \{0.7, 0.8\}$ and $B_{0.9}(0.4) \cap M = \{0.5\}$. Hence, 0.4 can be replaced by 0.5 (optimal central point of $\{0.5\}$) and 0.7 can be replaced by 0.75 (optimal central point of $\{0.7, 0.8\}$). As a result, we get a set $K' = \{0.5, 0.75\}$ for which $\mathrm{appr}(M, K') = 0.95 > 0.9 = \mathrm{appr}(M, K)$, i.e. K' provides a better approximation of M than K does.

Still, such improvement does not, in general, satisfy condition 3. of Problem 2. That is, replacement of points k in K by better points k' which cover the same part of M, i.e. for which $B_e(k) \cap M = B_e(k') \cap M$, does not result in the best approximating set with size $|K|$. An algorithm which provides such set, i.e. which provides a solution to Problem 3, is the subject of our future research.

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