Choquet integral and its modifications

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Abstract

In many decision problems a set of actions is evaluated with respect to a set of points of view, called criteria. The main aim of this contribution is to look for an appropriate utility function. Moreover, we will investigate the interactions of criteria, which have been considered through non-additive integrals such as Choquet integral. We illustrate our approach on a practical example.

Keywords: Choquet integral, fuzzy preference structures, Łukasiewicz filter, T_L evaluator

In many decision problems a set of actions is evaluated with respect to a set of points of view, called criteria. This paper follows two aims – first to compare the so-called level-dependent Choquet integral introduced recently by S. Greco. S. Giove and B. Matarazzo with other transformation of Choquet integral, proposed by Z. Havranová and M. Kalina. The other aim of this paper is to look for an appropriate utility function in a given setting. Moreover, we will investigate the interactions of criteria, which have been considered through non-additive integrals such as Choquet integral. We illustrate our approach on a practical example.

1 Preliminaries

Let us recall the well-known definitions.

Definition 1 *A triangular norm (t-norm for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms are satisfied:*
(T1) Commutativity

$$T(x,y) = T(y,x),$$

(T2) Associativity

$$T(x, T(y,z)) = T(T(x,y), z),$$

(T3) Monotonicity

$$T(x,y) \leq T(x,z) \quad \textit{whenever } y \leq z,$$

(T4) Boundary Condition

$$T(x,1) = x.$$

Remark 1. Note that, if T is a t-norm, then its dual t-conorm $S : [0,1]^2 \to [0,1]$ is given by

$$S(x,y) = 1 - T(1-x, 1-y).$$

In many areas of applications, comparison of elements from a given set is based on comparison of numerical evaluations of elements. Different types of evaluators, defined on some at most countable set X, have been used in our considerations.

Definition 2 *Let $X \neq \emptyset$ be a given at most countable set. Then a set-function $\mathcal{F} : 2^X \to [0,1]$ is said to be a Łukasiewicz filter on X if and only if the following are satisfied:*

1. $\mathcal{F}(X) = 1, \mathcal{F}(\emptyset) = 0$

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2. $(\forall A, B \subseteq X)\,(A \subseteq B \Rightarrow \mathcal{F}(A) \leq \mathcal{F}(B))$

3. *for all* A, $B \subseteq X$ *the following is satisfied*

$$\mathcal{F}(A \cap B) \geq T_L\,(\mathcal{F}(A), \mathcal{F}(B))\,.$$

Property 3 implies the following necessary condition, a function $\mathcal{F} : 2^X \to [0,1]$ must fulfil, to be a Łukasiewicz filter:

$$A \cap B = \emptyset \;\Rightarrow \mathcal{F}(A) + \mathcal{F}(B) \leq 1 \qquad (1)$$

for all A, $B \subseteq X$.

A complementary notion to Łukasiewicz filters is the notion of a Łukasiewicz ideal:

Definition 3 *Let $X \neq \emptyset$ be a given at most countable set. A set-function $\mathcal{I} : 2^X \to [0,1]$ is said to be a Łukasiewicz ideal on X iff:*

1. $\mathcal{I}(\emptyset) = 1,\ \mathcal{I}\,(X) = 0,$

2. *If $A \subseteq B \subseteq X$, then $\mathcal{I}(A) \geq \mathcal{I}(B)$,*

3. *for all* A, $B \subseteq X$ *the following is satisfied*

$$T_L\,(\mathcal{I}(A), \mathcal{I}(B)) \leq \mathcal{I}(A \cup B).$$

Let $X \neq \emptyset$ denote an, at most countable, fixed set. Then by $\mathcal{M}$ we denote the system of all functions $f : X \to [0,1]$. Hence $(\mathcal{M}, \wedge, \vee, \top, \bot)$ is a complete lattice with top and bottom elements $\top$ and $\bot$, equal to constants 1 and 0, respectively. D. Dubois, W. Ostasiewicz, H. Prade in [5] have characterized so-called evaluators defined by:

Definition 4 *A function $\varphi : \mathcal{M} \to [0,1]$ is called a normalized evaluator on $\mathcal{M}$ iff*

1. $\varphi(\top) = 1, \varphi(\bot) = 0,$

2. *for all f, $g \in \mathcal{M}$, if $f \leq g$ then $\varphi(f) \leq \varphi(g)$.*

In fact, normalized evaluators were defined for an arbitrary bounded lattice. However, we restrict our considerations to the lattice $\mathcal{M}$.

In [1] S. Bodjanova has proposed so-called T_L and S_L evaluators:

Definition 5 ([1]) *A normalized evaluator $\varphi : \mathcal{M} \to [0,1]$ is said to be a T_L evaluator on $\mathcal{M}$, if it satisfies the formula*

$$\varphi(f \wedge g) \geq T_L\,(\varphi(f), \varphi(g)) \qquad (2)$$

Definition 6 ([1]) *A normalized evaluator $\varphi : \mathcal{M} \to [0,1]$ is said to be an S_L evaluator on $\mathcal{M}$, if it satisfies the formula*

$$\varphi(f \vee g) \leq S_L\,(\varphi(f), \varphi(g))\,. \qquad (3)$$

Generalizing the results of [1], we get the following

Lemma 1 *Let $\varphi : \mathcal{M} \to [0,1]$ be a T_L evaluator. Then there exists a Łukasiewicz filter $\mathcal{F} : 2^X \to [0,1]$ such that for each $A \in 2^X$*

$$\mathcal{F}(A) = \varphi(1_A).$$

Lemma 2 *Let $\varphi : \mathcal{M} \to [0,1]$ be an S_L evaluator. Then there exists a Łukasiewicz ideal $\mathcal{I} : 2^X \to [0,1]$ such that for each $A \in 2^X$*

$$\mathcal{I}(A) = 1 - \varphi(1_A).$$

Łukasiewicz ultrafilters are some special Łukasiewicz filters:

Definition 7 *Let $X \neq \emptyset$ be the given set. A set-function $\mathcal{U} : 2^X \to [0,1]$ is said to be a Łukasiewicz ultrafilter on X iff*

1. *$\mathcal{U}$ is a Łukasiewicz filter on X*

2. *$\mathcal{I} = 1 - \mathcal{U}$ is a Łukasiewicz ideal on X.*

Łukasiewicz filters on X may be generalized to T_L evaluators (Lemma 1). Łukasiewicz ultrafilters on X may be generalized to evaluators which are T_L and S_L.

A preference structure is a basic concept of preference modelling. In a classical preference structure (PS), a decision-maker makes three decisions for any pair (a, b) from the set $\boldsymbol{A}$ of all alternatives. His or her decision defines a triplet P, I, J of crisp binary relations on $\boldsymbol{A}$:

1) a is preferred to $b \Leftrightarrow (a, b) \in P$ (strict preference).

2) a and b are indifferent $\Leftrightarrow$ $(a,b) \in I$ (indifference).

3) a and b are incomparable $\Leftrightarrow$ $(a,b) \in J$ (incomparability).

A preference structure (PS) on a set $\boldsymbol{A}$ is a triplet (P,I,J) of binary relations on $\boldsymbol{A}$ such that

(ps1) I is reflexive, P and J are antireflexive.

(ps2) P is asymmetric, I and J are symmetric.

(ps3) $P \cap I = P \cap J = I \cap J = \emptyset$.

(ps4) $P \cup I \cup J \cup P^t = \mathbf{A} \times \mathbf{A}$ where $P^t(x,y) = P(y,x)$.

A preference structure can be characterized by the reflexive relation $R = P \cup I$ called the large preference relation. It can be easily proved that

$$P = R \cap co(R^t), I = R \cap R^t, J = co(R) \cap co(R^t),$$

where $coR(a,b)$ is the complement of $R(a,b)$. This allows us to construct a preference structure (P,I,J) from a reflexive binary operation R only.

Decision-makers are often uncertain, even inconsistent, in their judgements. The restriction to two-valued relations has been an important drawback in their practical use. A natural demand led researchers to introducing of a fuzzy preference structure (FPS). The original idea of using numbers between zero and one to describe the strength of links between two alternatives goes back to Menger. The introducing of fuzzy relations enables us to express degrees of preference, indifference and incomparability. Of course, the attempts to simply replace the notion used in the definition of (PS) by their fuzzy equivalents have brought some problems.

To define (FPS) it is necessary to consider some fuzzy connectives. We shall consider a continuous De Morgan triple (T,S,N) consisting of a continuous t-norm T, continuous t-conorm S and a strong negator N such that $T(x,y) = N(S(N(x),N(y)))$. The main problem lies in the fact that the completeness condition (ps4) can be written in many forms, e.g.:

$$co(P \cup P^t) = I \cup J, P = co(P^t \cup I \cup J),$$

$$P \cup I = co(P^t \cup J).$$

Let (T,S,N) be a De Morgan triplet. A fuzzy preference structure (FPS) on a set $\boldsymbol{A}$ is a triplet (P,I,J) of binary fuzzy relations on $\boldsymbol{A}$ such that:

(f1) I is reflexive, P and J are antireflexive. $I(a,a) = 1$, $P(a,a) = J(a,a) = 0$

(f2) P is T-asymmetric, I and J are symmetric. $T(P(a,b),P(b,a)) = 0$

(f3) $T(P,I) = T(P,J) = T(I,J) = 0$ for all pair of alternatives

(f4) $(\forall(a,b) \in \mathbf{A})S(P,P^t,I,J) = 1$ or $N(S(P,I)) = S(P^t,J)$ or other completeness conditions.

In some cases of decision-making process it is important to get a T_L transitive fuzzy preference relation P. It means, if x,y,z are some objects, the following should hold:

$$T_L(P(x,y),P(y,z)) \leq P(x,z). \qquad (4)$$

If we have in each parameter a linear ordered set of values (which is usually the case), we may assume for each criterion

$$\min(P(x_i,y_i),P(y_i,z_i)) \leq P(x_i,z_i),$$

where $P(x_i,y_i)$ is the preference of x to y, restricted to the i-th criterion. If we use some T_L evaluator φ to aggregate partial preferences done parameter by parameter, we get exactly formula (4).

Hence we get the following theorem:

Theorem 1 *The relation P is T_L transitive if and only if the evaluator $\varphi : \mathcal{M} \to [0,1]$ is T_L.*

Moreover, the following can be proven:

Theorem 2 *Assume φ, restricted to functions possessing values 0 and 1 only, is a generalization of some Łukasiewicz ultrafilter. Then the relation J (relation of incomparability) is empty.*

This is the reason why we look for T_L evaluators. (More on the connection between Łukasiewicz filters and fuzzy preference relations the reader may find in [10].

The interaction of criteria has been considered through non-additive integrals such as Choquet integral. For the first time it was defined by Vitali (1925) and then by Choquet (1953-54, [4]). Further important paper on this kind of fuzzy integral was that of J. Šipoš [19] in 1979. We restrict our interest to the discrete case. The definition of the Choquet integral with respect to a fuzzy measure is the following:

Definition 8 *Let $\mu : 2^X \to [0,1]$ be a fuzzy measure, which is continuous from below. Then the following mapping, $C\int : \mathcal{M} \to [0,1]$, is called the Choquet integral with respect to μ:*

$$C\int f\, d\mu = \int_0^1 \mu(\{z; f(z) \geq x\})\, dx$$

for each $f \in \mathcal{M}$, where the right-hand integral is in the sense of Riemann.

Łukasiewicz filters are special fuzzy measures. I.e., we may integrate with respect to them. Choosing some particular Łukasiewicz filter means choosing weights of individual criteria, but also of any system of criteria (which means their interaction).

Example 1 Let us have three criteria: $C = \{c_1, c_2, c_3\}$. We choose a Łukasiewicz filter $\mathcal{F}$ in choosing weights for each criterion according to its importance, but also for each couple of criteria, see table 1. Of course, the complete set of criteria, C, has its weight equal to one. The system of weights has to fulfil conditions of Definition 2:

The connection between Łukasiewicz filters and Choquet integrals is given by the following two theorems:

Table 1: Table of weights

singletons	weights	couples	weights
c_1	0.1	c_1, c_2	0.5
c_2	0.3	c_1, c_3	0.7
c_3	0.5	c_2, c_3	0.9

Theorem 3 *Let $\mathcal{F}$ be an arbitrary Łukasiewicz filter on X. Then the Choquet integral with respect to $\mathcal{F}$ is a T_L evaluator.*

Theorem 4 *Let $\mathcal{U}$ be an arbitrary Łukasiewicz ultrafilter on X. Then the Choquet integral with respect to $\mathcal{U}$ is both T_L and S_L evaluator.*

2 Generalization of Choquet integral

With respect to classical Choquet integral, interaction indices have been introduced by Murofushi and Soneda ([17]) with respect to only couples of criteria and by Grabish ([6]) with respect to all possible subsets of criteria.

Definition 9 ([7]) *Let us consider a set of criteria $X = \{1, 2, ..., n\}$. We define a generalized capacity as function $\mu^G : 2^X \times [0,1] \to [0,1]$ such that*

1) for all $t \in [0,1]$ and $A \subset B \subset X$, $\mu^G(A,t) \leq \mu^G(B,t)$

2) for all $t \in [0,1], \mu^G(\emptyset, t) = 0$ and $\mu^G(X,t) = 1$

3) (The regularity property) *for all $t \in [0,1]$ and for all $A \subset X, \mu^G(A,t)$ is continuous with respect to t almost everywhere.*

Definition 10 ([7]) *We define the level-dependent Choquet integral of a function $f : X \to [0,1]$ with respect to the generalized capacity μ^G as follows:*

$$Ch^G(f, \mu^G) = \int_0^1 \mu^G(A(f,t), t)\, dt$$

where the right-hand-side is the Lebesgue integral and

$$A(f,t) = \{x \in X; f(x) \geq t\}.$$

Let us remark that the level-dependent Choquet integral can always be written as

$$Ch^G(f,\mu^G) = \sum_{i=1}^{n} \int_{x_{(i-1)}}^{x_i} \mu^G(A(f,t),t)\,dt.$$

Observe that condition 3) ensures that the level-dependent Choquet integral always exists. Further observe that it is no problem to generalize the level-dependent Choquet integral to the case when X is a countable set (and with additional property of measurability of μ^G even to uncountable set X).

Theorem 5 *Let $\mu^G : 2^X \times [0,1] \to [0,1]$ be a generalized capacity such that for each $t \in [0,1]$ $\mu^G(\cdot,t) : 2^X \to [0,1]$ is a Łukasiewicz filter. Then*

$$Ch^G(f,\mu^G) = \int_0^1 \mu^G(A(f,t),t)\,dt$$

is a T_L evaluator.

Theorem 6 *Let $\mu^G : 2^X \times [0,1] \to [0,1]$ be a generalized capacity such that for each $t \in [0,1]$ $\mu^G(\cdot,t) : 2^X \to [0,1]$ is a Łukasiewicz ultrafilter. Then*

$$Ch^G(f,\mu^G) = \int_0^1 \mu^G(A(f,t),t)\,dt$$

is a T_L and S_L evaluator.

Slightly modifying Theorem 4 from [10] we get the following result:

Theorem 7 *Let us denote for each $x \in X$ $\eta_x : [0,1] \to [0,1]$ some isotone transformation with 0 and 1 as fixed points. For each $f \in \mathcal{M}$ put $\tilde{\eta}(f)(x) = \eta_x(f(x))$. Further, let $\mathcal{F} : 2^X \to [0,1]$ be a Łukasiewicz filter. Then $\varphi : \mathcal{M} \to [0,1]$ defined by*

$$\varphi(f) = C\int \tilde{\eta}(f)\,d\mathcal{F} \tag{5}$$

is a T_L evaluator.

Theorem 8 *Let us denote for each $x \in X$ $\eta_x : [0,1] \to [0,1]$ some isotone transformation with 0 and 1 as fixed points. For each $f \in \mathcal{M}$ put $\tilde{\eta}(f)(x) = \eta_x(f(x))$. Further, let $\mathcal{U} : 2^X \to [0,1]$ be a Łukasiewicz ultrafilter. Then $\varphi : \mathcal{M} \to [0,1]$ defined by*

$$\varphi(f) = C\int \tilde{\eta}(f)\,d\mathcal{U} \tag{6}$$

is a T_L and S_L evaluator.

As the above theorems show, we have (at least) two different possibilities how to transform Choquet integral. Both of them lead under certain conditions to T_L evaluators. Yet we should show that these transformations are not equivalent:

- On one hand it is obvious that if we integrate a constant c using the generalized Choquet integral, the result is the same constant c. But if we transform somehow the values, the result need not be c.
- On the other hand, if we play with the 'level-wise' given generalized capacity μ^G, we may get a result which is not possible to model just transforming the values of the function to be integrated.

The transformation of values of f might be useful when, e.g., we discretize the scale of parameters. On the other hand, the level-dependent Choquet integral might be useful to model a situation when we would like to change the importance of criteria according the level they achieved.

3 Application

In this section we show a possible application of the level-dependent Choquet integral to the decision problem with interacting parameters. Assume the following simple example of selection of an appropriate flat for a young married couple, yet child-less.

Example 2 Let $X = \{x_1, x_2, \ldots, x_{50}\}$ be the set of 50 flats which we want to compare. For the simplicity, we restrict ourselves to the flats with two rooms. We have a set of 12 parameters (criteria) $K = \{k_1, k_2, \ldots k_{12}\}$, where k_1 is the area of a flat, k_2 is the distance to the city center, k_3 is the distance to the work, k_4

is the locality, k_5 is the price, k_6 are monthly expenses, k_7 is the necessary investment, k_8 is the age of a flat, k_9 is the existence of parking lot, k_{10} is the accessibility to public transport, k_{11} is the existence of a lift, k_{12} is the store. It is obvious that some criteria are interacting. Let us assume the following values and utility functions for criteria.

Table 2: k_1-area of a flat

low	medium	large
0	0.5	1

Table 3: k_2-distance to the city center

near	medium	far
1	0.75	0.25

Table 4: k_3-distance to the work

near	medium	far
1	0.75	0.25

Table 5: k_4-locality

city center	suburb A	suburb B	suburb C
1	0.5	0.5	0

k_5-price (in million of SKK)

$$f(x) = \begin{cases} 1 & \text{for } x \leq 1.6, \\ -\frac{10}{11}x + \frac{27}{11} & \text{for } x \in (1.6, 2.7), \\ 0 & \text{elsewhere.} \end{cases}$$

k_6-monthly expenses (in thousands of SKK)

$$f(x) = \begin{cases} 1 & \text{for } x \leq 2, \\ -\frac{1}{5}x + \frac{7}{5} & \text{for } x \in (2, 7), \\ 0 & \text{elsewhere.} \end{cases}$$

k_7-necessary investment (in thousands of SKK)

$$f(x) = \begin{cases} 1 & \text{for } x = 0, \\ -\frac{1}{300}x + 1 & \text{for } x \in (0, 300), \\ 0 & \text{elsewhere.} \end{cases}$$

k_8-the age of a flat (in years)

$$f(x) = \begin{cases} 0.75 & \text{for } x \in \langle 0, 10 \rangle, \\ 1 & \text{for } x \in (10, 15), \\ e^{-x+15} & \text{elsewhere.} \end{cases}$$

Table 6: k_9-the existence of a parkig lot

yes	no
1	0

Table 7: k_{10}-accessibility to public transport

good	medium	low
1	0.5	0

Table 8: k_{11}-the existence of a lift

yes	no
1	0

Table 9: k_{12}-store

$> 12, < 1$	8–11	2–3	4–7
0	0.25	1	0.75

The weights of criteria are listed in the following table. They were derived from the linguistic values, selected by the couple, {crucial, very important, important, partially important, slightly important}.

Table 10: Table of weights of parameters

Parameter	Weight
k_1	0.15
k_2	0.15
k_3	0.2
k_4	0.25
k_5	0.3
k_6	0.3
k_7	0.3
k_8	0.15
k_9	0.25
k_{10}	0.1
k_{11}	0.15
k_{12}	0.1

The importance of groups of criteria are given by a Łukasiewicz filter $\mathcal{F}$, we use to aggregate the weights. We will use special Łukasiewicz filters, so called (T_L, S)-filter (for details on (T_L, S)-filter see [14]). Particularly, for any subset $Z \subseteq X$ we get

$$\mathcal{F}(Z) = \begin{cases} 1, & \text{if } Z = X \\ S(z_i)_{z_i \in Z}, & \text{otherwise.} \end{cases}$$

The selected weights of criteria are not normalized. Their particular values depend on the t-conorm S. They must fulfil the necessary condition (1).

We put $S = S_M$ in the area where values are from the interval $[0, 0.25]$ and we put S equal to the ordinal sum $S = \{(]0, 0.2[, S_P), (]0.2, 1[, S_L)\}$ in the area where values are from the interval $[0.25, 1]$. Let us denote this generalized capacity $\mathcal{G}$.

For each criterion we can construct the table with fuzzy preferences. For simplicity, we restrict ourselves only to two flats. The fuzzy preference for a criterion k_i for flats x_1 and x_2 is computed from the utility functions as

$$FP_{k_1}(x_1, x_2) = \max\left\{\left(x_{k_1}^{x_1} - x_{k_1}^{x_2}\right), 0\right\},$$

where $x_{k_1}^{x_1}$ and $x_{k_1}^{x_2}$ are score of flats in criterion k_1.

	FP(x1,x2)	FP(x2,x1)
k_1	0.5	0
k_2	0	0.25
k_3	0	0
k_4	0.5	0
k_5	0	1
k_6	0	0
k_7	0	0
k_8	0	0.25
k_9	0	1
k_{10}	0	0
k_{11}	0	1
k_{12}	0.25	0

Then we have evaluated two functions:

$f_1 = k_1/0.5 + k_2/0 + k_3/0 + \cdots k_{12}/0.25$ and $f_2 = k_1/0 + k_2/0.25 + k_3/0 + \cdots k_{12}/0$.

$$C^G(f_1, \mathcal{G}) = 0.125,$$
$$C^G(f_2, \mathcal{G}) = 0.3375,$$

This means that we prefer the flat x_2 to x_1 at the level 0.3375.

It is obvious that the above mentioned procedure allows us to compare pairwise all flats in our data set under any selection of importance of criteria by our married couple.

4 Conclusion

In this paper we have compared two possible approaches to modification of Choquet integral. We have shown under which conditions these modifications lead to a T_L evaluator. Finally, we have applied modified Choquet integral to a decision-making problem with choosing an appropriate flat. In real situation this approach may be applied to choosing a flat from online available data set of a real-estate agency according to criteria given by the client.

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