Fuzzy envelope of a probability distribution

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Abstract

Let us consider a probability space, a random variable on it and the induced probability space. Now, let us consider a new random variable defined on this new space, and the new probability space induced by it. In this paper, we generalize this composition of randon variables to the case when the first variable is a fuzzy random variable instead of an ordinary random variable.

Keywords: Fuzzy random variable, random set, expectation, probability envelope.

1 Introduction

Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a random variable $U : \Omega \to \Omega'$. Now, we can consider the probability space $(\Omega', \mathcal{A}', P_U)$, where P_U is the probability induced by U. Let $(\Omega'', \mathcal{A}'')$ be another probability space, and let $X : (\Omega', \mathcal{A}', P_U) \to (\Omega'', \mathcal{A}'')$ be a random variable . So we can obtain the induced probability $P_U(X^{-1}(A))$ for all $A \in \mathcal{A}''$.

This concepts have been extended in [3] to the case we have a random set instead U, and its probability envelope instead P_U. In this paper, we are going to consider a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ and its probability envelope $P_{\tilde{X}}$. We will view a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ as the representation of the imprecise knowledge about the variable $U_0 : \Omega \to \Omega'$ (called "original" random variable).This is the possibilistic interpretation of fuzzy sets given by Kruse and Meyer ([12]). Therefore, the membership degree of a element $\omega' \in \Omega'$ to the fuzzy set $\tilde{X}(\omega)$ represents the acceptability degree of the assertion "$U_0(\omega) = \omega'$". This interpretation of a fuzzy random variable is associated with second order possibility distributions ([2, 4, 5, 6] among others).

In section 2 we review some concepts we will use throughout this work, for example the concept of fuzzy random variable and its probability envelope. In section 3 we extends the concept of the induced probability $P_U(X^{-1}(A))$ to the case we have a fuzzy random variable instead the classical random variable U. We obtain three different definitions and the relationship among them. In section 4 we do the same with the concept of expectation of the variable X. The extensions we build in this paper have a strong relationship with the original random variable and its probability and expectation.

2 Preliminary concepts

One of the most useful concepts we use in this paper is the concept of *graded set* ([11]). Graded sets allow us to prove our results in an easier way.

Definition 2.1 *([11]) For an arbitrary set, Ω, a graded set of Ω is a multi-valued mapping $\psi : [0,1] \to \mathcal{P}(\Omega)$ satisfaying*

$$\forall\, \alpha, \beta \in [0,1],\ [\alpha < \beta \Rightarrow \psi(\alpha) \supseteq \psi(\beta)]$$

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Therefore, a graded set is a nested family of subsets. An example of graded set are the α–cuts of a fuzzy set (strong and weak α–cuts), and there are at least 2 graded sets corresponding to each fuzzy set. On the other hand, for every graded set, there exists a unique fuzzy set corresponding to it given by:

$$\mu(\omega) = \sup\{\alpha \in [0,1] | \omega \in \psi(\alpha)\} \;\; \forall\, \omega \in \Omega$$

Let us now consider two measurable spaces, $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$. We will say that the multi-valued mapping $\Gamma : \Omega \to \mathcal{P}(\Omega')$ is *strongly measurable* ([14]) when:

$$\forall B \in \mathcal{A}' \quad \Gamma^*(B) := \{\omega \in \Omega | \Gamma(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$$

A strongly measurable mapping is also called *random set.*

A.P. Dempster ([7]) defines the concepts of upper and lower probabilities induced by a multi-valued mapping. We can consider the upper and lower probabilities induced by a random set, and we will denote them by P^* and P_*.

We will say that the mapping $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$, where $\widetilde{\mathcal{P}}(\Omega')$ denotes the power fuzzy set of ω', is a *fuzzy random variable* ([15]) when the multi-valued mapping $\tilde{X}_\alpha : \Omega \to \mathcal{P}(\Omega')$ defined by:

$$\tilde{X}_\alpha(\omega) := \left[\tilde{X}(\omega)\right]^{[\alpha]} \;\; \forall\, \omega \in \Omega$$

is strongly measurable $\forall \alpha \in [0,1]$. The multi-valued mapping $\tilde{X}_\alpha : \Omega \to \mathcal{P}(\Omega')$ is called *α–cut of* $\tilde{X}$. The fuzzy random variable and its α–cuts uniquely determine each other.

As we pointed in the introduction of the paper, we will view a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ as the representation of the imprecise knowledge about the variable $U_0 : \Omega \to \Omega'$ (called "original" random variable). Suppose we don't know the exact value of $U_0(\omega)$, $\omega \in \Omega$. Therefore, we consider the fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ such that the membership degree of a element $\omega' \in \Omega'$ to the fuzzy set $\tilde{X}(\omega)$ represents the possibility degree of the assertion "$U_0(\omega) = \omega'$".

Consider the set F of all measurable mappings $U : \Omega \to \Omega'$. We can build a fuzzy subset of F corresponding to $\tilde{X}$ as follows:

$$\mu_{\tilde{X}}(U) := \inf\{\tilde{X}(\omega)(U(\omega)) |\; \omega \in \Omega\}$$

The membership degree of a element $U \in F$ to the fuzzy set $\mu_{\tilde{X}}$ represents the possibility degree of the assertion "$U_0 = U$".

Given a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a random variable $U : \Omega \to \Omega'$, we can obtain the probability distribution induced by U. When we consider a fuzzy random variable, we build the *probability envelope*:

Definition 2.2 *([3, 6]) Let us consider a probability space $(\Omega, \mathcal{A}, P)$ and a measurable space $(\Omega', \mathcal{A}')$. Let $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ be a fuzzy random variable.*

The probability envelope induced by $\tilde{X}$ is defined as the mapping $P_{\tilde{X}} : \mathcal{A}' \to \widetilde{\mathcal{P}}([0,1])$ given by:

$$P_{\tilde{X}}(A')(p) := \sup_{\{U:\Omega\to\Omega' \;\; r.v. | P_U(A') = p\}} \mu_{\tilde{X}}(U)$$

Definition 2.3 *([9]) Let us consider a probability space $(\Omega, \mathcal{A}, P)$ and a measurable space $(\Omega', \mathcal{A}')$. Let $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ be a fuzzy random variable.*

For all $A' \in \mathcal{A}'$ we define the following quantities:

$$(P_*)_{\tilde{X}}(A') = \sup_{\alpha \in [0,1]} (P_*)_\alpha(A')$$

$$P^*_{\tilde{X}}(A') = \inf_{\alpha \in [0,1]} P^*_\alpha(A')$$

This quantities satisfy that

$$P_{U_0}(A') \in [(P_*)_{\tilde{X}}(A'), P^*_{\tilde{X}}(A')]$$

The last concept we will introduce is the expectation of a fuzzy random variable, defined as follows:

Definition 2.4 *([15]) Let $(\Omega, \mathcal{A}, P)$ be a probability space and let E be a Banach space. Given a fuzzy random variable $\tilde{X}$ from Ω to $\tilde{\mathcal{P}}(E)$, its expectation is a fuzzy set whose $\alpha-$cuts are given by the followings formula:*

$$[E(\tilde{X})]^{[\alpha]} = \int_{\Omega} \tilde{X}_{\alpha} dP, \ \alpha \in [0,1]$$

where $\int \cdot dP$ represents the Kudo–Aumann integral.

3 Fuzzy envelope of the probability induced by a random variable

Let us suppose we have a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a random variable $U : \Omega \to \Omega'$. We can build the probability space $(\Omega', \mathcal{A}', P_U)$.

The first definition we are going to see generalizes the concept of probability space induced by a random variable:

Definition 3.1 *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\Omega', \mathcal{A}')$ a measurable space. Let $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ be a fuzzy random variable. We call fuzzy probability space to $(\Omega', \mathcal{A}', P_{\tilde{X}})$, where $P_{\tilde{X}}$ is the probability envelope induced by $\tilde{X}$.*

Let us consider now a measurable space $(\Omega'', \mathcal{A}'')$ and let $X : \Omega' \to \Omega''$ be a random variable. We can obtain the probability distribution induced by X given by $P_{U_X}(A'') = P_U(X^{-1}(A''))$. The following definitions generalize this situation:

Definition 3.2 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$.*

We will call type-1 fuzzy envelope of the probability induced by X to the mapping $P^1_X : \mathcal{A}'' \to \tilde{\mathcal{P}}([0,1])$, where $P^1_X(A'')$ is a fuzzy set given by the following membership funcion:

$$P^1_X(A'')(p) := P_{\tilde{X}}(X^{-1}(A''))(p)$$

$$\forall p \in [0,1], \forall A'' \in \mathcal{A}''$$

If the fuzzy random variable represents the imprecise knowledge about a random variable $U_0 : \Omega \to \Omega'$, then the quantity $P^1_X(A'')(p)$ give us an aproximation of the acceptability degree of the assertion "$P_{U_0}(X^{-1}(A)) = p$".

Definition 3.3 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$.*

We will call type-2 fuzzy envelope of the probability induced by X to the mapping $P^2_X : \mathcal{A}'' \to \tilde{\mathcal{P}}([0,1])$, where

$$P^2_X(A'') = P_{X \circ \tilde{X}}(A'') \quad \forall A'' \in \mathcal{A}''$$

The fuzzy random variable $X \circ \tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega'')$ is defined as follows:

$$(X \circ \tilde{X})_{\alpha} = X \circ \tilde{X}_{\alpha}$$

and

$$X \circ \tilde{X}_{\alpha}(\omega) = \{X(\omega') | \omega' \in \tilde{X}_{\alpha}(\omega)\}$$

It is easy to see that if the fuzzy random variable $\tilde{X}$ represents the imprecise knowledge about $U_0 : \Omega \to \Omega'$, then the fuzzy random variable $X \circ \tilde{X}$ is the representation of the imprecise knowledge about $X \circ U_0$. Therefore, $P^2_X(A'')(p)$ give us an aproximation of the acceptability degree of the assertion "$P((X \circ U_0)^{-1}(A)) = p$".

Definition 3.4 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$.*

We will call type-3 fuzzy envelope of the probability induced by X to the mapping $P^3_X : \mathcal{A}'' \to \mathcal{P}([0,1])$ defined by the following formula:

$$P^3_X(A'') := [(P_*)_{X \circ \tilde{X}}(A''), (P^*)_{X \circ \tilde{X}}(A'')]$$

$$\forall A'' \in \mathcal{A}''$$

We easily observe that if the fuzzy random variable $\tilde{X}$ represents the imprecise knowledge about $U_0 : \Omega \to \Omega'$, then $P_{U_0}(X^{-1}(A'')) \in P_X^3(A'')$.

Since a random set is also a fuzzy random variable, we can obtain the type-i fuzzy envelope of the probability induced by it (i=1,2,3). The definitions given in this paper extends the concepts of envelope of the probability induced by a random set given in [3].

Let us examine now the relations between two different fuzzy envelope of the probability induced by X. In order to prove the relationship between type-1 and type-2, we will need the following lemmas:

Lemma 3.1 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$.*

Consider the type-1 fuzzy envelope of the probability induced by X, P_X^1, and the type-1 fuzzy envelope of the probability induced by its α–cuts $\tilde{X}_\alpha$ ($\alpha \in [0, 1]$) that will be denoted by $P_X^{1,\alpha}$.

Then, $\{P_X^{1,\alpha}(A'')\}_{\alpha\in[0,1]}$ is a graded set, and $P_X^1(A'')$ is the fuzzy set corresponding to it, $\forall A'' \in \mathcal{A}''$.

Lemma 3.2 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$.*

Consider the type-2 fuzzy envelope of the probability induced by X, P_X^2, and the type-2 fuzzy envelope of the probability induced by its α–cuts $\tilde{X}_\alpha$ ($\alpha \in [0, 1]$) that will be denoted by $P_X^{2,\alpha}$.

Then, $\{P_X^{2,\alpha}(A'')\}_{\alpha\in[0,1]}$ is a graded set, and $P_X^2(A'')$ is the fuzzy set corresponding to it, $\forall A'' \in \mathcal{A}''$.

The analogous result is not true when we consider the type-3 fuzzy envelope, due to $\{P_X^{3,\alpha}(A'')\}_{\alpha\in[0,1]}$ is not a graded set.

The relation between type-1 and type-2 fuzzy envelope of the probability induced by X is given by the following result:

Theorem 3.1 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$. Then*

$$P_X^1(A'') \subseteq P_X^2(A'') \quad \forall A'' \in \mathcal{A}''$$

Then, the aproximation given by the second envelope of the acceptability degree of the assertion "$P((X \circ U_0)^{-1}(A)) = p$" is better than the aproximation given by the fist envelope of the probability induced by X, $\forall p \in [0, 1]$.

The relationship between the second and the third fuzzy envelope is detailed below. The proof is based on interpretation given for the probability envelope induce by a fuzzy random variable.

Theorem 3.2 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$. Let us consider a measurable space $(\Omega'', \mathcal{A}'')$ and a random variable $X : \Omega' \to \Omega''$. Then*

$$P_X^2(A'') \subseteq P_X^3(A'') \quad \forall A'' \in \mathcal{A}''$$

In conclusion,

$$P_X^1(A'') \subseteq P_X^2(A'') \subseteq P_X^3(A'') \quad \forall A'' \in \mathcal{A}''$$

This inclusions can be strict, since this definitions extend the concepts given in [3] and we can find there some examples of that assertion.

4 Fuzzy envelope of the expectation

Let us suppose we have a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a random variable $U : \Omega \to \Omega'$. We can build the probability space $(\Omega', \mathcal{A}', P_U)$. Now let us consider a random variable $X : \Omega' \to \mathbb{R}^m$. We can obtain the expectation of X, $E_{P_U}(X)$. In this section, we have a fuzzy random variable instead U. We will give several definitions for the concept of fuzzy envelope of the expectation of a random variable, and we will examine the relationship among them.

Definition 4.1 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \tilde{\mathcal{P}}(\Omega')$. Given a random variable $X : \Omega' \to \mathbb{R}^m$ we will call type-1 fuzzy envelope of the expectation of X to the fuzzy set $E^1_{P_{\tilde{X}}}(X)$, whose membership function is given by the following formula:*

$$E^1_{P_{\tilde{X}}}(X)(\vec{a}) := \sup_{\{\vec{a}=E_{P_U}(X), U \ r.v.,\ \mu_{\tilde{X}}(U)\geq\alpha\}} \alpha$$

Definition 4.2 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \tilde{\mathcal{P}}(\Omega')$. Given a random variable $X : \Omega' \to \mathbb{R}^m$ we will call type-2 fuzzy envelope of the expectation of X to the fuzzy set*

$$E^2_{P_{\tilde{X}}}(X) = E(X \circ \tilde{X})$$

The quantities $E^1_{P_{\tilde{X}}}(X)(\vec{a})$ and $E^2_{P_{\tilde{X}}}(X)(\vec{a})$ gives us an aproximation of the acceptability degree of the assertion "$E_{P_{U_0}}(X) = \vec{a}$", if $\tilde{X}$ represents the imprecise knowledge about U_0.

For obtaining the following definitions, we have considered the following result:

Lemma 4.1 *Let $(\Omega, \mathcal{A}, P)$ a probability space, and $X : \Omega \to \mathbb{R}$ a random variable. Then:*

$$E(X) = \int_0^\infty P(X > t)dt - \int_{-\infty}^0 P(X < t)dt$$

Definition 4.3 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \tilde{\mathcal{P}}(\Omega')$. Let us consider a random variable $X : \Omega' \to \mathbb{R}$ and the following mappings: $Z : \mathbb{R} \to \tilde{\mathcal{P}}(\mathbb{R})$, $Z(t) = P_{\tilde{X}}\left(X^{-1}(t, \infty)\right)$ and $G : \mathbb{R} \to \tilde{\mathcal{P}}(\mathbb{R})$, $G = Z \cdot I_{[0,\infty)} + (Z - 1) \cdot I_{(-\infty,0)}$.*

We will call type-3 fuzzy envelope of the expectation of X to the fuzzy set

$$E^3_{P_{\tilde{X}}}(X) = \int_{\mathbb{R}} G d\lambda$$

where $[\int_{\mathbb{R}} G d\lambda]^{[\alpha]} = \int_{\mathbb{R}} G_\alpha d\lambda \ \forall \alpha \in [0, 1]$, the Kudo–Aumann integral of the random set G_α.

Definition 4.4 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \tilde{\mathcal{P}}(\Omega')$. Let us consider a random variable $X : \Omega' \to \mathbb{R}$ and the following mappings: $Z' : \mathbb{R} \to \tilde{\mathcal{P}}(\mathbb{R})$, $Z'(t) = P_{X\circ\tilde{X}}(t, \infty)$ and $G' : \mathbb{R} \to \tilde{\mathcal{P}}(\mathbb{R})$, $G' = Z' \cdot I_{[0,\infty)} + (Z' - 1) \cdot I_{(-\infty,0)}$.*

We will call type-4 fuzzy envelope of the expectation of X to the fuzzy set

$$E^4_{P_{\tilde{X}}}(X) = \int_{\mathbb{R}} G' d\lambda$$

where $[\int_{\mathbb{R}} G' d\lambda]^{[\alpha]} = \int_{\mathbb{R}} G'_\alpha d\lambda \ \forall \alpha \in [0, 1]$, the Kudo–Aumann integral of the random set G'_α.

Since a random set is also a fuzzy random variable, we can obtain the type-i fuzzy envelope of the expectation (i=1,2,3) of a random variable X. The definitions given in this paper extends the concepts of envelope of the expectation of X given in [3].

Lemma 4.2 *Let us consider a probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(\Omega', \mathcal{A}')$ and a fuzzy random variable $\tilde{X} : \Omega \to \tilde{\mathcal{P}}(\Omega')$. Let $X : \Omega' \to \mathbb{R}^m$ be a random variable.*

Consider the type-1 fuzzy envelope of the expectation of X, $E^1_{P_{\tilde{X}}}(X)$, and the type-1 fuzzy envelope of the probability induced by its $\alpha-$cuts $\tilde{X}_\alpha$ ($\alpha \in [0, 1]$) which will be denoted by $E^1_{P_{\tilde{X}_\alpha}}(X)$.

Then, $\{E^1_{P_{\tilde{X}_\alpha}}(X)\}_{\alpha\in[0,1]}$ is a graded set, and $E^1_{P_{\tilde{X}}}(X)$ is the fuzzy set corresponding to it.

Lemma 4.3 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$*, a measurable space* $(\Omega', \mathcal{A}')$ *and a fuzzy random variable* $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$*. Let* $X : \Omega' \to \mathbb{R}^m$ *be a random variable.*

Consider the type-2 fuzzy envelope of the expectation of X*,* $E^2_{P_{\tilde{X}}}(X)$*, and the type-2 fuzzy envelope of the probability induced by its* $\alpha-$*cuts* $\tilde{X}_\alpha$ $(\alpha \in [0,1])$ *which will be denoted by* $E^2_{P_{\tilde{X}_\alpha}}(X)$*.*

Then, $\{E^2_{P_{\tilde{X}_\alpha}}(X)\}_{\alpha\in[0,1]}$ *is a graded set, and* $E^2_{P_{\tilde{X}}}(X)$ *is the fuzzy set corresponding to it.*

Proposition 4.1 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$ *and a measurable space* $(\Omega', \mathcal{A}')$*. Let* $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$ *be a fuzzy random variable and* $X : \Omega' \to \mathbb{R}^m$ *a random variable. Then*

$$E^1_{P_{\tilde{X}}}(X) \subseteq E^2_{P_{\tilde{X}}}(X)$$

Lemma 4.4 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$*, a measurable space* $(\Omega', \mathcal{A}')$ *and a fuzzy random variable* $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$*. Let* $X : \Omega' \to \mathbb{R}$ *be a random variable.*

Consider the type-3 fuzzy envelope of the expectation of X*,* $E^3_{P_{\tilde{X}}}(X)$*, and the type-3 fuzzy envelope of the probability induced by its* $\alpha-$*cuts* $\tilde{X}_\alpha$ $(\alpha \in [0,1])$ *that will be denoted by* $E^3_{P_{\tilde{X}_\alpha}}(X)$*.*

Then, $E^3_{P_{\tilde{X}_\alpha}}(X) \subseteq [E^3_{P_{\tilde{X}}}(X)]^{[\alpha]}$ $\forall \alpha \in [0,1]$*.*

Lemma 4.5 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$*, a measurable space* $(\Omega', \mathcal{A}')$ *and a fuzzy random variable* $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$*. Let* $X : \Omega' \to \mathbb{R}$ *be a random variable.*

Consider the type-4 fuzzy envelope of the expectation of X*,* $E^4_{P_{\tilde{X}}}(X)$*, and the type-4 fuzzy envelope of the probability induced by its* $\alpha-$*cuts* $\tilde{X}_\alpha$ $(\alpha \in [0,1])$ *that will be denoted by* $E^4_{P_{\tilde{X}_\alpha}}(X)$*.*

Then, $E^4_{P_{\tilde{X}_\alpha}}(X) \subseteq [E^4_{P_{\tilde{X}}}(X)]^{[\alpha]}$ $\forall \alpha \in [0,1]$*.*

Theorem 4.1 *Let us consider a probability space* $(\Omega, \mathcal{A}, P)$*, a measurable space* $(\Omega', \mathcal{A}')$ *and a fuzzy random variable* $\tilde{X} : \Omega \to \widetilde{\mathcal{P}}(\Omega')$*. Let* $X : \Omega' \to \mathbb{R}$ *be a random variable. Then:*

- $E^1_{P_{\tilde{X}}}(X) \subseteq E^2_{P_{\tilde{X}}}(X)$
- $E^1_{P_{\tilde{X}}}(X) \subseteq E^3_{P_{\tilde{X}}}(X)$
- $E^2_{P_{\tilde{X}}}(X) \subseteq E^4_{P_{\tilde{X}}}(X)$
- $E^3_{P_{\tilde{X}}}(X) \subseteq E^4_{P_{\tilde{X}}}(X)$

5 Conclusion

In this paper, we have extended the concepts of probability induced by a random variable X and its expectation when we consider de fuzzy probability space $(\Omega, \mathcal{A}, P_{\tilde{X}})$ where $P_{\tilde{X}}$ is the probability envelope induced by the fuzzy random variable $\tilde{X}$. We have considered a fuzzy random variable as the representation of the imprecise knowledge about the "original" random variable U_0. We have been able to give several definitions of the concept of fuzzy envelope of the probability induced by a random variable, and we have obtained the relationship among them. For example, we have prove that the type-2 fuzzy envelope is better aproximation of the probability we are interesting in: $(P_{U_0})_X(A'') = P_{U_0}(X^{-1}(A''))$ for all $A'' \in \mathcal{A}''$. With respect to the fuzzy envelope of the expectation, we have also obtain several definitions, and we have compared them. In a future work, we will obtain new definitions of the fuzzy envelope of the expectation and we will compare them with the definitions given in this paper. Moreover, we will study another properties of the different types of envelope of the expectation, such us lineality.

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