On the polytope of fuzzy measures

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Abstract

In this paper we deal with the problem of studying some mathematical aspects regarding the polytope of fuzzy measures when the referential set is finite. More concretely, we study whether two extreme points are adjacent; for general polytopes, this is a NP-hard problem. However, for the case of fuzzy measures, we give a necessary and sufficient condition for two extreme points to be adjacent. This allows us to prove that it is possible to find out in polynomial time (in the number of minimal subsets) whether two extremes are adjacent. These results can be extended to the polytope given by the convex hull of monotone boolean functions.

Keywords: Fuzzy measures, monotone boolean functions, adjacency, complexity.

1 Introduction and basic concepts

Consider a finite referential set $X = \{x_1, ..., x_n\}$ of n elements. The set X is the set of criteria in Decision Making, players in Game Theory, individuals in Welfare Theory, and so on. Subsets of X are denoted by capital letters $A, B, ...$ In order to avoid hard notation, for singletons $\{x_i\}$ we will usually omit braces. The set of subsets of X is denoted by $\mathcal{P}(X)$.

Definition 1. *A* **non-additive measure** *[8] or* **fuzzy measure** *[25] or* **capacity** *[4] over X is a function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying*

1. $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$.
2. *If* $A \subseteq B$ *then* $\mu(A) \leq \mu(B)$.

From a mathematical point of view, fuzzy measures constitute a generalization of probability distributions and additive weights in which we remove additivity and monotonicity is imposed instead. This extension is perfectly justified in many practical situations, in which additivity is too restrictive. Fuzzy measures have been applied to many fields, as Decision Making under Risk and Uncertainty [2, 24], Multicriteria Decision Making [13, 14], pseudo-boolean functions [16], etc. This versatility of fuzzy measures has led to a huge number of related works, both from a theoretical and from a practical point of view.

We will denote the set of all fuzzy measures over X by $\mathcal{FM}(X)$. Notice that $\mathcal{FM}(X)$ is a polytope in $\mathbb{R}^{2^n-2}$ (or $\mathbb{R}^{2^n}$ if we include the coordinates for $\mu(\emptyset)$ and $\mu(X)$).

On $\mathcal{FM}(X)$ we can define a partial order given by $\mu_1 \leq \mu_2$ if and only if $\mu_1(A) \leq \mu_2(A), \forall A \subseteq X$. If $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$ we say that μ_1 and μ_2 are **comparable**.

On $\mathcal{FM}(X)$ we can also define the following operations:

- The **supremum** $(\mu_1 \vee \mu_2)(A) := \max(\mu_1(A), \mu_2(A)), \forall A \subseteq X$.
- The **infimum** $(\mu_1 \wedge \mu_2)(A) := \min(\mu_1(A), \mu_2(A)), \forall A \subseteq X$.

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 71–78
Torremolinos (Málaga), June 22–27, 2008

Note that we need $2^n - 2$ coefficients in order to define a fuzzy measure. Then, additional constraints are introduced in order to reduce the complexity. This has led to several subfamilies of fuzzy measures, as p-symmetric measures, k-intolerant measures, λ-measures and so on. One of the most important families of fuzzy measures is the family of k-additive measures. The concept of k-additivity is based on the Möbius transform.

Definition 2. *[23] Let μ be a set function (not necessarily a fuzzy measure) on X. The* **Möbius transform (or inverse)** *of μ is another set function on X defined by*

$$m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \, \forall A \subseteq X.$$

The Möbius transform given, the original set function can be recovered through the *Zeta transform* [3]:

$$\mu(A) = \sum_{B \subseteq A} m(B).$$

The Möbius transform represents the importance that a subset can attain on its own, without considering its different parts.

Definition 3. *[12] A fuzzy measure μ is said to be k-***additive** *if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.*

These measures constitute a bridge between probabilities and general fuzzy measures. We will denote by $\mathcal{FM}^k(X)$ the polytope of all fuzzy measures on X being at most k-additive.

A problem arising in practice is the identification of the fuzzy measure modelling a certain situation. In [5], we have dealt with the problem of identifying a fuzzy measure from sample information through genetic algorithms [11]. The cross-over operator used in the algorithm was the convex combination, possible as $\mathcal{FM}(X)$ is a polytope; this operator allows a reduction in the complexity of the algorithm. However, the use of this operator has the drawback that the search region is reduced in each iteration. Then, in order to ensure that the searched measure is inside the region, we need to consider the extreme points of $\mathcal{FM}(X)$ as the initial population. It has been pointed out in [21] that these extreme points are the set of $\{0,1\}$-valued measures, i.e. the set of monotone boolean functions of n variables except the constant functions 0 and 1.

In a previous paper [19], we have proved that, unexpectedly, there are vertices of $\mathcal{FM}^k(X), k > 2$ that are not $\{0,1\}$-valued measures. These vertices are convex combinations of $\{0,1\}$-valued measures that determine a r-dimensional facet of $\mathcal{FM}(X)$. Therefore, it seems interesting to study the adjacency in $\mathcal{FM}(X)$.

Remark that for a $\{0,1\}$-valued measure μ, there are some subsets A satisfying the following conditions:

$$\begin{array}{l} \mu(A) = 1, \\ \mu(B) = 1, \quad \forall B \supseteq A, \\ \mu(C) = 0, \quad \forall C \subset A. \end{array}$$

We will call any subset satisfying these conditions a **minimal subset** for μ. The set of minimal subsets also forms the *qualitative Möbius transform* [15]. Note that a $\{0,1\}$-valued measure is completely defined by its minimal subsets. To see this, it suffices to remark that $\mu(A) = 1$ if A contains a minimal subset and $\mu(A) = 0$ otherwise. We will denote the families of minimal subsets by $\mathbf{C}, \mathbf{D}$, and so on. The fuzzy measure whose minimal subsets are the family $\mathbf{C}$ will be denoted by $\mu_{\mathbf{C}}$.

An example of fuzzy measure that we will use later is the one in which there is only a minimal subset $A \subseteq X$, $A \neq \emptyset$. This measure is given by

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}$$

These fuzzy measures are the vertices of a special convex class of fuzzy measures called *belief functions*, that appear in the Theory of Evidence [7]. For $\emptyset$, we define the fuzzy measure $u_\emptyset$ by

$$u_\emptyset(B) := \begin{cases} 1 & \text{if B} \neq \emptyset \\ 0 & \text{if B} = \emptyset \end{cases}$$

Note that $u_\emptyset$ follows a different structure to any other u_A (indeed it is not a belief function).

The set of minimal subsets of a $\{0,1\}$-valued measure determine an **antichain** (collections of sets which are pairwise uncomparable with respect to inclusion, see [1]). Then, the number of vertices of the polytope $\mathcal{FM}(X)$ is the number of different antichains on X. The number of antichains of a set of cardinality n is known as the *n-th Dedekind number* [6]. The first Dedekind numbers are given in Table 1.

Table 1: Number of vertices of $\mathcal{FM}(X)$

n	Dedekind numbers
1	1
2	4
3	18
4	166
5	7579
6	7828352
7	2414682040996
8	56130437228687557907786

In the values given in this table we have excluded the empty antichain and the antichain which contains only the empty set, as these cases do not lead to a fuzzy measure (they do not satisfy the boundary conditions). The form of the general term of this sequence is not known and, in fact, only few of them have been calculated up today. However, their values are bounded:

Theorem 1. *[9] If D_n is the n-th Dedekind number and $E_n := D_n + 2$, then it holds*

$$2^q \leq D_n, E_n \leq 3^q,$$

with $q = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $\lfloor x \rfloor$ the integer part of x, for all $n \geq 1$.

The paper is organized as follows: In next section we study when two extreme points of $\mathcal{FM}(X)$ are adjacent. Then, in Section 3 we show that we can find out whether two vertices are adjacent in polynomial time. These properties could be interesting in the problem of identification of fuzzy measures. Moreover, they might shed light on the structure of $\mathcal{FM}(X)$. We also prove that these results can be extended to the convex hull of monotone boolean functions. We finish with the conclusions and open problems.

2 Adjacency on $\mathcal{FM}(X)$

Let us address the problem of whether two $\{0,1\}$-valued measures are adjacent. Remark that two vertices of a polytope are adjacent if the middle point of the segment joining them cannot be written as a convex combination of other vertices of the polytope. In this case, the segment is, in fact, an edge of the polytope. Consider a polytope of the form $\{A\vec{x} \leq \vec{b}\}$ where $\vec{x} \in \mathbb{R}^p$ and suppose there are not dummy constraints. Given two vertices $\vec{x}, \vec{y}$, let us denote by B the submatrix of A consisting in the constraints that any point $\vec{z}$ in the segment joining $\vec{x}, \vec{y}$ excluded $\vec{x}$ and $\vec{y}$ satisfies with equality; then, it is well-known that $\vec{x}, \vec{y}$ are adjacent if the rank of B is $p-1$.

In the case of $\mathcal{FM}(X)$, these constraints are

$$\mu(\emptyset) = 0, \mu(X) = 1.$$

$$\mu(A) - \mu(A \backslash x_i) \geq 0 \quad \forall A \subseteq X, x_i \in A \quad (1)$$

Then, we have 2 equalities and $n2^{n-1}$ inequalities. Thus, given two vertices μ_1, μ_2, they are adjacent if the range of the constraints in Eq. (1) that $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies with equality is $2^n - 3$ (or $2^n - 1$ if we include the boundary conditions). Note that the constraints that $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies with equality are exactly the constraints that both μ_1 and μ_2 satisfy with equality. This provides us with an algorithm to determine whether two vertices are adjacent. However, it is very slow. Indeed, it can be proved that the complexity can grow in a non-polynomial way.

As proved in [17, 20], the problem of determining non-adjacency of vertices of a polytope is, in some cases, NP-complete (see [10] for a definition of NP-complete problems and related notions). However, in this section we give a necessary and sufficient condition of adjacency on $\mathcal{FM}(X)$ which can be used to prove that the problem of determining adjacency can be solved, in this case, in polynomial time in the number of minimal subsets of the vertices (see Theorem 5 below).

Notice that for general boolean functions (not necessarily monotone), the problem of adjacency is easy to solve: It can be proved that two boolean functions are adjacent if and only if their Hamming distance (the number of inputs on which they differ) is 1.

Definition 4. *Given a polytope $\mathcal{F}$, the* **graph of** $\mathcal{F}$ *is defined by the graph whose nodes are the vertices of $\mathcal{F}$ and whose edges join two nodes if and only if they are adjacent.*

It can be seen that the Hamming distance between two boolean functions is equal to the length of the shortest path between them in the graph of adjacency. In particular, if the distance of two monotone boolean functions is 1, then they are adjacent. However, the converse is not true; for instance, $u_{x_1} \wedge u_{x_2}$ and u_{x_1} are adjacent as monotone boolean functions although their distance is greater than 1 when $n > 2$ (they differ, at least, in $\{x_1\}$ and in $\{x_1, x_3\}$). In fact, we will show below that $u_{x_1} \vee u_{x_2} \vee \ldots \vee u_{x_n}$ and $u_{x_1} \wedge u_{x_2} \wedge \ldots \wedge u_{x_n}$ are adjacent when $n > 2$ (see Theorem 2), although, as proved in [18], their distance is the maximum possible distance between two extremes of fuzzy measures.

The following result is obvious.

Lemma 1. *For $\mu_1, \mu_2 \in \mathcal{FM}(X)$, $\mu_1 \wedge \mu_2$ and $\mu_1 \vee \mu_2$ satisfy*

$$\frac{1}{2}(\mu_1 + \mu_2) = \frac{1}{2}((\mu_1 \wedge \mu_2) + (\mu_1 \vee \mu_2))$$

and $\mu_1 \wedge \mu_2 \leq \mu_1, \mu_2 \leq \mu_1 \vee \mu_2$.

Corollary 1. *If μ_1 and μ_2 are adjacent vertices of $\mathcal{FM}(X)$, then $\mu_1 < \mu_2$ or $\mu_2 < \mu_1$.*

Proof: Suppose μ_1 and μ_2 are not comparable. Then, $\mu_1 \wedge \mu_2 \neq \mu_1, \mu_2$ and the same holds for $\mu_1 \vee \mu_2$. By Lemma 1, this implies that μ_1 and μ_2 are not adjacent. ■

However, this is not a sufficient condition. For instance, assume $|X| \geq 2$ and consider $\mu = u_{\{x_i, x_j\}}$ and μ' defined by

$$\mu'(A) := \begin{cases} 1 & \text{if } x_i \in A \text{ or } x_j \in A \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mu < \mu'$ but they are not adjacent as

$$\frac{1}{2}(u_{\{x_i, x_j\}} + \mu') = \frac{1}{2}(u_{x_i} + u_{x_j}).$$

In fact, if two vertices μ_1 and μ_2 are not comparable, it follows from Lemma 1 that their supremum and infimum are not adjacent, though they are always comparable.

In next definition we introduce the concept of **C**-decomposability, that will be needed in the following results.

Definition 5. *Let* **C** *and* **D** *be two collections of minimal sets (antichains). We say that* **D** *is* **C−decomposable** *if there exists a partition of* **D** *in two non-empty subsets* **A** *and* **B** *such that $\mathbf{A} \not\subseteq \mathbf{C}$ and $\mathbf{B} \not\subseteq \mathbf{C}$, and if $A \in \mathbf{A}$ and $B \in \mathbf{B}$, then there exists $C \in \mathbf{C}$ such that $C \subseteq A \cup B$.*

The idea of **C**-decomposability is the following: consider two fuzzy measures $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ such that $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$. If **D** is **C**-decomposable, then it is possible to obtain two other antichains **A** and **B** from which, together with a partition of **C**, we can derive two fuzzy measures μ_1 and μ_2 satisfying

$$\frac{1}{2}(\mu_{\mathbf{D}} + \mu_{\mathbf{C}}) = \frac{1}{2}(\mu_1 + \mu_2).$$

This result is stated in the next result, whose proof is rather technical.

Theorem 2. *Let $\mu_{\mathbf{D}}, \mu_{\mathbf{C}}$ be two vertices of $\mathcal{FM}(X)$ such that $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$. Then, $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ are adjacent vertices of $\mathcal{FM}(X)$ if and only if* **D** *is not* **C***-decomposable.*

Let us give two examples of how this result can be applied:

Example 1. *For $\emptyset \neq A, B \subseteq X$, consider the corresponding unanimity games u_A and u_B. As an application of Theorem 2, we conclude that u_A and u_B are adjacent in $\mathcal{FM}(X)$ if and only if $A \subset B$ or $B \subset A$. From Corollary 1 it follows that $A \subset B$ or $B \subset A$ is a necessary condition. Also, the collection of minimal sets of these unanimity games consists only of one set, so no decomposition is possible and from Theorem 2 it follows that the condition is also sufficient. Notice, however, that the result is not necessarily true if one of the sets A or*

B is empty, since the minimal sets of $u_\emptyset$ *are* $\{\{x_1\}, \ldots, \{x_n\}\}$.

Moreover, if μ *is a vertex of* $\mathcal{FM}(X)$ *satisfying that* $\mu < u_A$ $(A \neq \emptyset)$ *then* μ *and* u_A *are adjacent.*

Example 2. *If* $|X| > 2$ *then* $u_\emptyset$ *and* u_X *are adjacent since the minimal sets of* $u_\emptyset$ *are* $\{\{x_1\}, \ldots, \{x_n\}\}$ *and they cannot be* $\{X\}$*-decomposed when* $|X| > 2$*. However, when* $|X| = 2$ *we have*

$$\frac{1}{2}(u_X + u_\emptyset) = \frac{1}{2}(u_{x_1} + u_{x_2})$$

and then $u_\emptyset$ *and* u_X *are not adjacent.*

Notice, however, that $u_\emptyset \geq \mu \geq u_X, \forall \mu \in \mathcal{FM}(X)$*. Thus,* $d(u_\emptyset, u_X)$ *is maximal in* $\mathcal{FM}(X)$.

This last example is generalized in the following Proposition in order to cover the cases not considered in Example 1.

Proposition 1. *If* $E \neq \emptyset$ *then* u_E *and* $u_\emptyset$ *are not adjacent if and only if* $|X| = 2$ *and* $E = X$.

Figure 1 (which has been drawn with the help of the Pigale computer program[1]) depicts the adjacency graph for $|X| = 3$. We represent each vertex by means of its minimal sets. Also, we use i instead of x_i. Thus, $\{1, 2\}, \{3\}$ stands for $(u_{x_1} \wedge u_{x_2}) \vee u_{x_3}$, and so on.

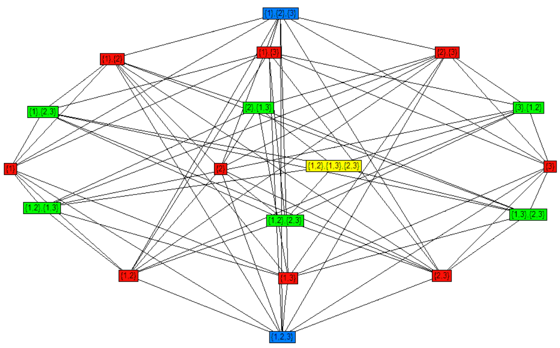

Figure 1: Adjacency of vertices of $\mathcal{FM}(X)$ for $|X| = 3$

From the proof of Theorem 2, it is also possible to derive the following results:

[1] PIGALE: Public Implementation of a Graph Algorithm Library and Editor, H. de Fraysseix and P. Ossona de Mendez. http://pigale.sourceforge.net/

Corollary 2. *Given two extreme points* μ, μ' *of* $\mathcal{FM}(X)$ *that are not adjacent, then there exist* μ_1, μ_2 *other extreme points of* $\mathcal{FM}(X)$ *such that*

$$\frac{1}{2}(\mu + \mu') = \frac{1}{2}(\mu_1 + \mu_2).$$

Of course, if μ, μ' are not adjacent, then $\frac{1}{2}(\mu + \mu')$ can be written as a convex combination of other vertices. What is interesting in this result is that it can be put as a convex combination of exactly two vertices, both of them with weight $\frac{1}{2}$.

Corollary 3. *Let* μ *and* μ' *be two vertices of* $\mathcal{FM}(X)$*. Suppose the midpoint between* μ *and* μ' *can be written as a convex combination of some vertices of* $\mathcal{FM}(X)$*, namely,*

$$\frac{1}{2}(\mu + \mu') = \sum_{i=1}^{k} \lambda_i \mu_i,$$

with $\lambda_i > 0, \forall i = 1, ..., k$ *and* $\sum_{i=1}^{k} \lambda_i = 1$*. Then, for each* $i = 1, \ldots, k$*, we have that either* $\mu_i = \mu, \mu_i = \mu'$*, or there exists* μ'_i *vertex of* $\mathcal{FM}(X)$ *such that* $\frac{1}{2}(\mu + \mu') = \frac{1}{2}(\mu_i + \mu'_i)$.

This result gives a sufficient and necessary condition for a vertex to appear in a convex combination which is equal to the midpoint of a segment. If μ and μ' are adjacent vertices, then each μ_i is either μ or μ'. The second part of the result only applies when μ and μ' are not adjacent. Notice, however, that in this last case, the measure μ'_i is not necessarily one of the measures $\mu_1, ..., \mu_j$ as next example shows:

Example 3. *For* $|X| = 3$, *consider*

$$\mu = (u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3}) \vee (u_{x_2} \wedge u_{x_3})$$

$$\mu' = u_{x_1} \wedge u_{x_2} \wedge u_{x_3}.$$

Then,

$$\frac{1}{2}(\mu + \mu') = \frac{1}{4}\{[(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})] + [(u_{x_1} \wedge u_{x_2}) \vee (u_{x_2} \wedge u_{x_3})] + [(u_{x_1} \wedge u_{x_3}) \vee (u_{x_2} \wedge u_{x_3})] + (u_{x_1} \wedge u_{x_2} \wedge u_{x_3})\}.$$

For $(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})$*, we obtain that* $\frac{1}{2}(\mu + \mu')$ *is equal to*

$$\frac{1}{2}\{[(u_{x_1} \wedge u_{x_2}) \vee (u_{x_1} \wedge u_{x_3})] + (u_{x_2} \wedge u_{x_3})\}.$$

However, when only two vertices $\mu_{\mathbf{E_1}}$ and $\mu_{\mathbf{E_2}}$ are involved in a convex combination which is equal to the midpoint of other two vertices $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$, then it is always the case that $\mu_{\mathbf{E_1}}$ and $\mu_{\mathbf{E_2}}$ come from a **C**-decomposition of **D** as the following Theorem shows.

Theorem 3. *Suppose $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ are two extreme points of $\mathcal{FM}(X)$ such that $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$ and they are not adjacent. Consider $\mu_{\mathbf{E_1}}$ and $\mu_{\mathbf{E_2}}$ two vertices of $\mathcal{FM}(X)$ different from $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ such that $\frac{1}{2}(\mu_{\mathbf{D}}+\mu_{\mathbf{C}}) = \frac{1}{2}(\mu_{\mathbf{E_1}}+\mu_{\mathbf{E_2}})$. Then, $\mathbf{A} = \mathbf{E_1} \cap \mathbf{D}$ and $\mathbf{B} = (\mathbf{E_2} \cap \mathbf{D}) \setminus \mathbf{A}$ form a **C**-decomposition of **D**.*

When dealing with monotone Boolean functions, it is usual to include the constant functions 0 and 1, which are not fuzzy measures. For these functions, the following can be shown:

Lemma 2. *The constant functions 0 and 1 are adjacent to any other monotone Boolean function.*

Remark that the collections of minimal subsets for 0 and 1 are respectively $\emptyset$ and $\{\emptyset\}$. This allows us to extend Theorem 2 for any monotone Boolean function.

Corollary 4. *Let $\mu_{\mathbf{D}}, \mu_{\mathbf{C}}$ be two vertices of the set of the convex hull of monotone Boolean functions such that $\mu_{\mathbf{D}} > \mu_{\mathbf{C}}$. Then, $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{C}}$ are adjacent vertices of this polytope if and only if $\mu_{\mathbf{D}}$ is not **C**-decomposable.*

Similarly, Corollaries 2 and 3 also apply to the convex hull of monotone Boolean functions.

3 Complexity of determining adjacency in $\mathcal{FM}(X)$

Let us now study the complexity of determining whether two $\{0,1\}$-valued measures are adjacent. Consider two antichains **C** and **D** in X corresponding to the minimal subsets of two extreme points of $\mathcal{FM}(X)$. If these measures are not comparable, then we already know that they are not adjacent by Corollary 1. Let us then assume that $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$.

Definition 6. *Let **C** and **D** be two collections of subsets of X. We associate to them the graph $G_{\mathbf{C},\mathbf{D}}$ whose nodes are the elements of $\mathbf{D} \setminus \mathbf{C}$ and such that there is an edge between A and B if there is no $C \in \mathbf{C}$ such that $C \subseteq A \cup B$.*

Example 4. *Consider $|X| = 4$ and $\mathbf{D} = \{(1,2),(1,3),(1,4),(2,3),(2,4)\}, \mathbf{C} = \{(1,2)\}$. Then, the associated graph $G_{\mathbf{C},\mathbf{D}}$ is given in Figure 2.*

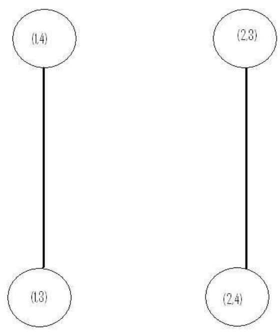

Figure 2: Example of graph $G_{\mathbf{C},\mathbf{D}}$

From this definition, it is easy to prove the following result.

Theorem 4. *Let $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ be two vertices of $\mathcal{FM}(X)$ such that $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$. Then $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ are adjacent if and only if the graph $G_{\mathbf{C},\mathbf{D}}$ is connected.*

Moreover, we can obtain as a consequence of this result the number of different decompositions. This gives us the number of different pairs of vertices μ_1, μ_2 satisfying

$$\frac{1}{2}(\mu_{\mathbf{C}} + \mu_{\mathbf{D}}) = \frac{1}{2}(\mu_1 + \mu_2).$$

Corollary 5. *Let $\mu_{\mathbf{C}}$ and $\mu_{\mathbf{D}}$ be two vertices of $\mathcal{FM}(X)$ such that $\mu_{\mathbf{C}} < \mu_{\mathbf{D}}$ and they are not adjacent. If c is the number of connected components of $G_{\mathbf{C},\mathbf{D}}$ and $d = |\mathbf{D} \cap \mathbf{C}|$, then the number of different **C**-decompositions of **D** is $(2^c - 2)2^{d-1}$.*

Finally, from Theorem 4 we can check whether two vertices are adjacent in polynomial time.

Theorem 5. *Let μ and μ' two vertices of $\mathcal{FM}(X)$. We can check whether they are adjacent in polynomial time in the number of* *minimal subsets of μ plus the number of minimal subsets of μ'.*

Proof: We can check whether $\mu < \mu'$ (or $\mu > \mu'$) in polynomial time in the number of minimal subsets of the extremes. If they are not comparable, then they are not adjacent. Otherwise, we can construct their associated graph (Definition 6) in polynomial time and check whether it is connected or not (for instance by breadth-first search, see [22]) also in polynomial time. The result follows then from Theorem 4. ∎

As the monotone Boolean functions 0 and 1 are adjacent to any other monotone Boolean function, the following can be deduced:

Corollary 6. *Let μ and μ' be two vertices of the convex hull of monotone Boolean functions. We can check whether they are adjacent in polynomial time in the number of minimal subsets of μ plus the number of minimal subsets of μ'.*

4 Conclusions and open problems

In this paper we have studied some properties of the polytope $\mathcal{FM}(X)$. These results apply also to the convex hull of monotone Boolean functions.

We have characterized from a mathematical point of view whether two vertices of $\mathcal{FM}(X)$ are adjacent. Moreover, we have proved that given two vertices, it can be computed in polynomial time if they are adjacent. We have also found all the possibilities of expressing the sum of two vertices of $\mathcal{FM}(X)$ that are not adjacent as sum of two other vertices.

We think that these results can shed light on the structure of $\mathcal{FM}(X)$ and the convex hull of monotone Boolean functions. Moreover, these results could be interesting in the problem of identifying a non-additive measure. For example, if we consider the identification through genetic algorithms, we know that the set of vertices cannot be used as the initial population [5]. However, it could be interesting to study the performance of the algorithm if we consider a subset of vertices such that two vertices in this initial population are not adjacent to each other.

When dealing with fuzzy measures, we are usually restricted to a subset of $\mathcal{FM}(X)$ satisfying an additional property. An interesting subfamily of non-additive measures are the so-called k-additive measures, where k varies in $\{1, ..., n-1\}$ [14]. As explained in the introduction, the results obtained in this paper could help us to study the structure of the vertices of this family.

It would be also interesting to study the relationship of the techniques introduced in this paper to explore the adjacency of vertices with the usual methods of linear programming (in [19] we have fruitfully applied linear programming tools to derive some results about the structure of k-additive measures).

Acknowledgements

This research has been supported in part by grant numbers MTM2007-61193 and CAM-UCM910707, and by MEC and FEDER grants TIN2004-05920 and TIN2007-61273.

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