Cubical Transition Systems

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Abstract

The paper deals with combinatorial and stochastic structures of cubical token systems. A cubical token system is an instance of a token system, which in turn is an instance of a transition system. A formal theory based on a system of four independent axioms for cubical token systems and main algebraic properties of these systems are introduced. A representation theorem for a cubical token system is established asserting that the graph of such a system is cubical. Stationary distributions for random walks on cubical token systems are calculated.

Keywords: Token system, cubical system, medium, Markov chain

1 Introduction

Cubical token systems and media are particular instances of a general algebraic structure, called 'token system', describing a mathematical, physical, or behavioral system as it evolves from one 'state' to another. This structure is formalized as a pair $(\mathcal{S}, \mathcal{T})$ consisting of a set $\mathcal{S}$ of states and a set $\mathcal{T}$ of tokens. Tokens are transformations of the set of states. Strings of tokens are 'messages' of the token system.

In the field of computer science, tokens systems are special forms of 'transition systems' [19]. However, we do not follow this lead in the paper. Instead, we propose a system of axioms specifying a class of token systems. In the framework of axiomatic approach, we make no assumptions regarding the nature of states and tokens. Four independent axioms are postulated on the pair $(\mathcal{S}, \mathcal{T})$, which then called a *cubical token system*. The name is justified by the result of Section 6 asserting that the graph of a cubical system is cubical, that is, a subgraph of a hypercube.

The concept of a medium was introduced in [5] as a token system specified by another system of constraining axioms, and developed further in [6, 16]. For more recent advances in media theory the reader is referred to [13, 14] and the monograph "Media Theory" [4].

While the concepts of cubical token systems and media may seem abstract and far remote from applications, they are easily specialized into concrete models, offering useful tools for the analysis of data. Two examples are selected to illustrate instances of cubical token systems.

(1) *Preference relations.* In many empirical situations in social sciences, subjects are repeatedly asked to provide judgments concerning commodities or individuals. It is typical that these judgments take the form of binary relations such as weak orders, semiorders, or other partial orders. A family $\mathcal{S}$ of partial orders on a finite set, equipped with the set of transformations $\mathcal{T}$ consisting in adding (or removing) an ordered pair to (or from) a partial order to form another partial order in $\mathcal{S}$ is an instance of a token system. The token system $(\mathcal{S}, \mathcal{T})$ is cubical in many applications

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(see examples in Section 7).

(2) *Learning spaces.* The knowledge structure [3] is a family $\mathcal{K}$ of subsets of a basic set Q of items of knowledge. Each of the sets in $\mathcal{K}$ is a (knowledge) state, representing the competence of a particular individual in the population of reference. In a typical example, the set Q is a set of questions (problems) each of which has a correct response. The knowledge state of an individual is represented by the set of questions in Q that the individual is capable of answering in ideal conditions. A learning space is a knowledge structure satisfying two compelling axioms. To cast a learning space as a token system $(\mathcal{K}, \mathcal{T})$, one takes any knowledge state to be a state of the token system; the transformations in $\mathcal{T}$ consist in adding or removing an item to (from) a state. The token system $(\mathcal{K}, \mathcal{T})$ is cubical. Furthermore, it is a medium. The computer educational system ALEKS provides on the Internet (`www.aleks.com`) an educational environment based on the theory of learning spaces. This system is currently used by many schools in the US and abroad.

We begin by introducing basic concepts of token systems in Section 2 and axioms for cubical token systems and media in Section 3, where it is also shown that media form a proper subclass of cubical systems.

G-systems are token systems defined on connected families of sets. In Section 4 we show that they are instances of cubical systems. G-systems are typical examples of cubical systems as it is demonstrated in Section 6.

Structural properties of states and messages of a cubical system are established in Section 5 in terms of their 'contents'. These properties are crucial for the development of stochastic token theory presented in Section 8, where stochastic properties of cubical token systems are established.

The main result of the algebraic part of cubical systems theory—the representation theorem—is established in Section 6 (Theorem 6.1). In Section 7 we give examples of cubical systems.

Proofs of the results presented in the paper are found in [15].

2 Token systems

In this section we introduce basic concepts of token systems theory.

Let $\mathcal{S}$ be a set of *states*. A *token* is a transformation $\tau : S \mapsto S\tau$. By definition, the identity function τ_0 on $\mathcal{S}$ is not a token. Let $\mathcal{T}$ be a set of tokens. The pair $(\mathcal{S}, \mathcal{T})$ is called a *token system* [5]. To avoid trivialities, we assume that $|\mathcal{S}| \geq 2$ and $\mathcal{T} \neq \varnothing$.

Let S and V be two states of a token system $(\mathcal{S}, \mathcal{T})$. Then V is *adjacent* to S if $V \neq S$ and $V = S\tau$ for some token $\tau \in \mathcal{T}$. A token $\tilde{\tau}$ is a *reverse* of a token τ if, for all distinct $S, V \in \mathcal{S}$, we have

$$S\tau = V \quad \Longleftrightarrow \quad V\tilde{\tau} = S.$$

Two distinct states S and V are *adjacent* if S is adjacent to V and V is adjacent to S.

A *message* of a token system $(\mathcal{S}, \mathcal{T})$ is a string of elements of the set $\mathcal{T}$. We write these strings in the form $\boldsymbol{m} = \tau_1\tau_2\ldots\tau_n$. If a token τ occurs in the string $\tau_1\tau_2\ldots\tau_n$, we say that the message $\boldsymbol{m} = \tau_1\tau_2\ldots\tau_n$ *contains* τ.

A message $\boldsymbol{m} = \tau_1\tau_2\ldots\tau_n$ defines a transformation

$$S \mapsto S\boldsymbol{m} = ((\ldots((S\tau_1)\tau_2)\ldots)\tau_n)$$

of the set of states $\mathcal{S}$. By definition, the empty message defines the identity transformation τ_0 of $\mathcal{S}$. If $V = S\boldsymbol{m}$ for some message $\boldsymbol{m}$ and states $S, V \in \mathcal{S}$, then we say that $\boldsymbol{m}$ *produces V from S* or, equivalently, that $\boldsymbol{m}$ *transforms S into V*. More generally, if $\boldsymbol{m} = \tau_1\ldots\tau_n$, then we say that $\boldsymbol{m}$ *produces a sequence of states* (S_i), where $S_i = S\tau_0\tau_1\ldots\tau_i$.

If $\boldsymbol{m}$ and $\boldsymbol{n}$ are two messages, then $\boldsymbol{mn}$ stands for the concatenation of the strings $\boldsymbol{m}$ and $\boldsymbol{n}$. We denote by $\widetilde{\boldsymbol{m}} = \tilde{\tau}_n\ldots\tilde{\tau}_1$ the *reverse* of the message $\boldsymbol{m} = \tau_1\ldots\tau_n$, provided that the tokens in $\widetilde{\boldsymbol{m}}$ exist.

A message $\boldsymbol{m} = \tau_1\ldots\tau_n$ is *vacuous* if the set of indices $\{1,\ldots,n\}$ can be partitioned into pairs i, j with $i \neq j$, such that τ_i and τ_j are mutual reverses.

A message $\boldsymbol{m}$ is *effective* (respectively *ineffective*) for a state S if $S\boldsymbol{m} \neq S$ (respectively $S\boldsymbol{m} = S$) for the corresponding transformation $\boldsymbol{m}$. A message $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is *stepwise effective* for S if $S_i \neq S_{i-1}$, $1 \leq k \leq n$, in the sequence of states (S_i) produced by $\boldsymbol{m}$ from S. A message is *closed* for a state S if it is stepwise effective and ineffective for S.

Two token systems $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}', \mathcal{T}')$ are said to be *isomorphic* if there is a pair (α, β) of bijections $\alpha : \mathcal{S} \to \mathcal{S}'$ and $\beta : \mathcal{T} \to \mathcal{T}'$ such that

$$S\tau = T \quad \Leftrightarrow \quad \alpha(S)\beta(\tau) = \alpha(T)$$

for all $S, T \in \mathcal{S}$ and $\tau \in \mathcal{T}$.

3 Axioms for cubical systems and media

Definition 3.1. A token system $(\mathcal{S}, \mathcal{T})$ is called a *cubical token system* if the following axioms are satisfied:

[C1] Every token $\tau \in \mathcal{T}$ has a reverse $\tilde{\tau} \neq \tau$ in $\mathcal{T}$.

[C2] For any two distinct states S and T there is a stepwise effective message producing T from S.

[C3] A message which is stepwise effective for some state is closed for that state if and only if it is vacuous.

[C4] If $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is a stepwise effective message for some state, then occurrences of a token and its reverse alternate in $\boldsymbol{m}$. More specifically, if $\tau_i = \tau_j = \tau$ for $i < j$ and some $\tau \in \mathcal{T}$, then $\tau_k = \tilde{\tau}$ for some $i < k < j$.

In the rest of the paper, we leave out the word "token" in "cubical token system".

Theorem 3.1. *Axioms* [C1]–[C4] *are independent.*

We need the concept of a 'concise message' for the definition of a medium.

Definition 3.2. A message $\boldsymbol{m}$ is said to be *concise* for a state S if: (i) $\boldsymbol{m}$ is stepwise effective for S, (ii) no token occurs twice in $\boldsymbol{m}$, and (iii) $\boldsymbol{m}$ does not contain a token and its reverse.

Definition 3.3. A token system $(\mathcal{S}, \mathcal{T})$ is called a *medium* (on $\mathcal{S}$) if the following axioms are satisfied:

[Ma] For any two distinct states S and V in $\mathcal{S}$ there is a concise message transforming S into V.

[Mb] A message which is closed for some state is vacuous.

Theorem 3.2. *A medium is a cubical system.*

As the following example demonstrates, the class of media is a proper subclass of cubical systems.

Example 3.1. Let $(\mathcal{S}, \mathcal{T})$ be a token system displayed in Figure 3.1. There is no concise message producing P from S, so this token system is not a medium. It is easy to verify that this system is a cubical system.

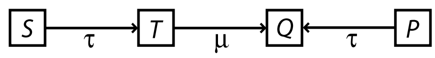

Figure 3.1: Token system $(\mathcal{S}, \mathcal{T})$ with $\mathcal{S} = \{S, T, P, Q\}$ and $\mathcal{T} = \{\tau, \tilde{\tau}, \mu, \tilde{\mu}\}$.

4 A 'canonical' example of a cubical system

A 'canonical' example of a medium is the representing medium of a well-graded family of sets (see Definition 7.1) [6, 14, 16]. For cubical systems, similar examples are given by G-systems.

Definition 4.1. An *embedding* of a graph $G = (V, E)$ into a graph $G' = (V', E')$ is an edge preserving map $V \to V'$. In this case we also say that G is *embeddable* into G'. A *cube* $\mathcal{H}(X)$ *on a set* X is a graph that has the set of all finite subsets of X as the set of vertices; $\{S, T\}$ is an edge of $\mathcal{H}(X)$ if $|S \triangle T| = 1$. A graph is said to be *cubical* if it is embeddable into a cube.

Definition 4.2. Let $G = (\mathcal{F}, \mathcal{E})$ be a connected subgraph of the cube $\mathcal{H}(X)$ on a set X with $|X| \geq 2$. A *G-system on* $\mathcal{F}$ is a pair $(\mathcal{F}, \mathcal{T}_G)$ where $\mathcal{T}_G$ is a family of transformations $\{\gamma_x, \tilde{\gamma}_x\}$ defined by

$$S\gamma_x = \begin{cases} S \cup \{x\}, & \text{if } \{S, S \cup \{x\}\} \in \mathcal{E}, \\ S, & \text{otherwise,} \end{cases}$$

$$S\tilde{\gamma}_x = \begin{cases} S \setminus \{x\}, & \text{if } \{S, S \setminus \{x\}\} \in \mathcal{E}, \\ S, & \text{otherwise,} \end{cases}$$

for all $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$.

Theorem 4.1. *A G-system on $\mathcal{F}$ is a token system and, for any $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$, the tokens γ_x and $\tilde{\gamma}_x$ are mutual reverses.*

To show that G-systems are cubical, we begin with a simple observation. Let $S_0 = S, S_1, \ldots, S_n = T$ be a walk in G. For an edge $\{S_{i-1}, S_i\}$, we denote $\{x_i\} = S_{i-1} \triangle S_i$, $\tau_i = \gamma_{x_i}$ if $S_i = S_{i-1} \cup \{x_i\}$, and $\tau_i = \tilde{\gamma}_{x_i}$, otherwise. Then $\boldsymbol{m} = \tau_1 \ldots \tau_n$ is a stepwise effective message for S of the G-system $(\mathcal{F}, \mathcal{T}_G)$. Conversely, a stepwise effective message $\boldsymbol{m} = \tau_1 \ldots \tau_n$ of $(\mathcal{F}, \mathcal{T}_G)$ producing a state T from a state S defines a walk $W_{\boldsymbol{m}}$ in G with vertices $S_i = S\tau_0\tau_1 \ldots \tau_n$. Thus there is a one-to-one correspondence between the stepwise effective messages of a G-system and the walks in G.

Theorem 4.2. *A G-system on $\mathcal{F}$ is a cubical system on the set of states $\mathcal{F}$.*

We will show in Section 6 (Theorem 6.1) that any cubical system is isomorphic to a G-system. Thus, G-systems are 'typical' instances of cubical systems.

5 Tokens and contents

Tokens of a cubical system share many properties with tokens of a medium (cf. Lemmas 5.1 and 5.2 in [14]).

Lemma 5.1. *The following statements hold for a cubical system $(\mathcal{S}, \mathcal{T})$:*

(i) $\tilde{\tilde{\tau}} = \tau$ *for any* $\tau \in \mathcal{T}$.

(ii) *For any two adjacent states S and T there is a unique token producing T from S.*

(iii) *If S, T, and P are three distinct states such that $S\tau = T$ and $T\mu = P$, for some tokens τ and μ, then $\mu \neq \tau$ and $\mu \neq \tilde{\tau}$.*

(iv) *No token can be a one-to-one function.*

Remark 5.1. Property (ii) of Lemma 5.1 is a very strong property of tokens of a cubical system. It asserts that two tokens τ and μ transforming some state S into a different state T are equal transformations, that is, $V\tau = V\mu$ for all $V \in \mathcal{S}$.

Let τ be a token of a medium. We define

$$\mathcal{U}_\tau = \{S \in \mathcal{S} \mid S\tau \neq S\}.$$

Note that $\mathcal{U}_\tau \neq \varnothing$, since τ is a token.

Lemma 5.2. *Let $(\mathcal{S}, \mathcal{T})$ be a cubical systems. For any given $\tau \in \mathcal{T}$ we have*

(i) $(\mathcal{U}_\tau)\tau = \mathcal{U}_{\tilde{\tau}}$.

(ii) $\mathcal{U}_\tau \cap \mathcal{U}_{\tilde{\tau}} = \varnothing$.

(iii) *The restriction $\tau|_{\mathcal{U}_\tau}$ is a bijection from $\mathcal{U}_\tau$ onto $\mathcal{U}_{\tilde{\tau}}$ with $\tau|_{\mathcal{U}_\tau}^{-1} = \tilde{\tau}|_{\mathcal{U}_{\tilde{\tau}}}$.*

Definition 5.1. Let $(\mathcal{S}, \mathcal{T})$ be a cubical system. For any token τ and any message $\boldsymbol{m}$, we define $\#(\tau, \boldsymbol{m})$ as the number of occurrences of τ in $\boldsymbol{m}$. For any message $\boldsymbol{m}$, the *content* of $\boldsymbol{m}$ is the set $\mathcal{C}(\boldsymbol{m})$ defined by

$$\mathcal{C}(\boldsymbol{m}) = \{\tau \in \mathcal{T} \mid \#(\tau, \boldsymbol{m}) > \#(\tilde{\tau}, \boldsymbol{m})\}.$$

For any state S, the *content* $\widehat{S}$ of S is the union $\cup_{\boldsymbol{m}}\mathcal{C}(\boldsymbol{m})$ taken over the set of all stepwise effective messages producing the state S.

The two concepts of 'content' are different from their counterparts in media theory. For instance, the content of a vacuous message of a cubical system is empty, whereas it is not empty in media theory. However, the main results of media theory concerning these concepts are valid for cubical systems. We establish these results in a series of theorems in the rest of this section where we assume that a cubical system $(\mathcal{S}, \mathcal{T})$ is given. Note that the results of Theorems 5.2 and 5.4 are especially useful in the stochastic part of cubical systems theory (Section 8).

Lemma 5.3. *If* $\boldsymbol{m}$ *is a stepwise effective message for some state, then*

$$\tau \in \mathcal{C}(\boldsymbol{m}) \quad \Leftrightarrow \quad \#(\tau, \boldsymbol{m}) = \#(\tilde{\tau}, \boldsymbol{m}) + 1.$$

Therefore, for any $\tau \in \mathcal{T}$,

$$\#(\tau, \boldsymbol{m}) - \#(\tilde{\tau}, \boldsymbol{m}) \in \{-1, 0, 1\}.$$

Lemma 5.4. *The content of a state cannot contain both a token and its reverse.*

Theorem 5.1. *For any token* τ *and any state* S *of a cubical system, we have either* $\tau \in \widehat{S}$ *or* $\tilde{\tau} \in \widehat{S}$ *(but not both).*

Theorem 5.2. *If* S *and* V *are two distinct states, with* $S\boldsymbol{m} = V$ *for some stepwise effective message* $\boldsymbol{m}$*, then* $\widehat{V} \setminus \widehat{S} = \mathcal{C}(\boldsymbol{m})$*. Therefore,*

$$\widehat{S} \triangle \widehat{V} = \mathcal{C}(\boldsymbol{m}) + \mathcal{C}(\widetilde{\boldsymbol{m}}),$$

where $+$ *stands for the disjoint union of two sets. In particular,*

$$\widehat{S} \triangle \widehat{V} = \{\tau, \tilde{\tau}\},$$

if $S\tau = V$.

Lemma 5.5. *A stepwise effective message* $\boldsymbol{m}$ *is closed if and only if*

$$\mathcal{C}(\boldsymbol{m}) = \varnothing.$$

Theorem 5.3. *For any two states* S *and* V *we have*

$$\widehat{S} = \widehat{V} \quad \Leftrightarrow \quad S = T.$$

Theorem 5.4. *Let* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *be two stepwise effective messages transforming some state* S*. Then*

$$S\boldsymbol{m} = S\boldsymbol{n} \quad \Leftrightarrow \quad \mathcal{C}(\boldsymbol{m}) = \mathcal{C}(\boldsymbol{n}).$$

We conclude this section by comparing two concepts of contents with their counterparts in media theory. The two statements of the next theorem assert that: (a) the content of a concise message of a medium is the same as its content in the medium, and (b) contents of the states of a medium are the same as defined in media theory.

Theorem 5.5. (i) *If* $\boldsymbol{m} = \tau_1 \ldots \tau_n$ *is a concise message of a medium, then*

$$\mathcal{C}(\boldsymbol{m}) = \{\tau_1, \ldots, \tau_n\}.$$

(ii) *For any state* S *of a medium, its content* $\widehat{S}$ *is the set of all tokens each of which is contained in at least one concise message producing* S.

6 A representation theorem for cubical systems

Definition 6.1. The *graph* G of a cubical system $(\mathcal{S}, \mathcal{T})$ has $\mathcal{S}$ as the set of its vertices; two vertices are adjacent in G if the corresponding states are adjacent in $(\mathcal{S}, \mathcal{T})$.

Theorem 6.1. *Let* $(\mathcal{S}, \mathcal{T})$ *be a cubical system. There exists a connected subgraph* $G = (\mathcal{F}, \mathcal{E})$ *of some cube* $\mathcal{H}(X)$ *such that* $(\mathcal{S}, \mathcal{T})$ *is isomorphic to the* G*-system* $(\mathcal{F}, \mathcal{T}_G)$ *on the family* $\mathcal{F}$.

Since cubical systems $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{F}, \mathcal{T}_G)$ of Theorem 6.1 are isomorphic, their graphs are isomorphic to the graph G.

Theorem 6.2. *The graph of a cubical system is cubical. Conversely, any cubical graph* G *defines a cubical system (a* G*-system).*

7 Examples of cubical token systems

We begin by introducing a class of finite G-systems that serves as a source of our examples (cf. [1]).

Definition 7.1. Let $\mathcal{F}$ be a family of subsets of a finite set X with $|\mathcal{F}| \geq 2$. A set $S \in \mathcal{F}$ is said to be *downgradable* if there exists $x \in S$ such that $S \setminus \{x\} \in \mathcal{F}$. The family $\mathcal{F}$ itself is *downgradable* if all its nonminimal sets are downgradable. Likewise, a set $S \in \mathcal{F}$ is said to be *upgradable* if there exists $x \in X \setminus S$ such that $S \cup \{x\} \in \mathcal{F}$. The family $\mathcal{F}$ itself is *upgradable* if all but its maximal sets are upgradable. We say that the family $\mathcal{F}$ is *well-graded* if the induced subgraph $\langle \mathcal{F} \rangle$ of the cube $\mathcal{H}(X)$ is an isometric subgraph of $\mathcal{H}(X)$ (cf. [2]).

It is clear that any downgradable family $\mathcal{F}$ of sets containing the empty set is connected, that is, for any two distinct sets

$S, V \in \mathcal{F}$ there is a sequence of sets $S_0 = S, S_1, \ldots, S_n = V$ such that $|S_i \triangle S_{i+1}| = 1$. Likewise, any upgradable family of subsets of X containing the set X itself is connected. Let $\mathcal{F}$ be any of such families. Then the induced subgraph $\langle \mathcal{F} \rangle$ of the cube $\mathcal{H}(X)$ is connected and therefore defines a cubical system (an $\langle \mathcal{F} \rangle$-system).

Example 7.1. *Comparability Graphs.* A simple finite graph $G = (X, E)$ is called a *comparability graph* [8] if there exists a partial order P on X such that

$$xy \in E \Leftrightarrow (x, y) \in P \text{ or } (y, x) \in P \quad (7.1)$$

We denote $\mathcal{CG}$ the family of all comparability graphs on a fixed set X and identify this family with the family of all sets of edges of these graphs. Clearly, $\mathcal{CG}$ contains the empty graph on X. It is known (see, for instance, [2]) that the family $\mathcal{PO}$ of all partial orders on X is well-graded. As it can be easily seen this fact implies that the family $\mathcal{CG}$ is downgradable and therefore defines a cubical system.

Figure 7.1: Two comparability graphs with 6 and 8 edges, respectively. The distance between the two edge sets is 2. There is no comparability graph on distance 1 from each of these two graphs. Thus the family $\mathcal{CG}$ is not well-graded.

Note that the wellgradedness property of the family $\mathcal{PO}$ does not imply that $\mathcal{CG}$ is well-graded (see the graphs in Figure 7.1).

Example 7.2. *Interval Graphs.* Interval and indifference graphs [8] are complements of comparability graphs arising from interval orders and semiorders, respectively, via relation (7.1). As the families of all interval orders and all semiorders are well-graded [2] and both contain the empty relation, the respective families of interval and indifference graphs are upgradable and both contain the complete graph on X. Thus we can cast each of these two families as a cubical system.

Note that the same result holds for any family of indifference graphs associated with partial orders satisfying so-called "distinguishing property" [12].

8 Stochastic cubical token systems

For basic definitions and results of Markov chain theory the reader is referred to textbooks [7, 10]. Following [5] we consider a discrete stochastic process arising when random events result in occurrences of tokens in a countable (or finite) cubical system $(\mathcal{S}, \mathcal{T})$.

Definition 8.1. A quadruple $(\mathcal{S}, \mathcal{T}, \xi, \theta)$ is a *probabilistic cubical token system* if the following three conditions hold:

(i) $(\mathcal{S}, \mathcal{T})$ is a cubical system.

(ii) $\xi : S \mapsto \xi(S)$ is a probability distribution (the *initial distribution*) on $\mathcal{S}$.

(iii) $\theta : \tau \mapsto \theta_\tau$ is a probability distribution on $\mathcal{T}$ with $\theta_\tau > 0$ for all tokens τ in $\mathcal{T}$.

Selecting an initial state according to the distribution ξ, and applying occurring tokens first to the initial state and then to its images under successive tokens, we obtain a Markov chain which we denote by $(\mathbf{S}_n)$, where n is the number of trials. The transition matrix $\mathbf{P}$ of this chain is given by the equations

$$p(S, V) = \begin{cases} \theta_\tau & \text{if } S\tau = V, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } V \neq S,$$

and

$$p(S, S) = 1 - \sum_{V \in \mathcal{S} \setminus \{S\}} p(S, V).$$

Note that $0 < p(S, S) < 1$, since, by Axiom [C2], for any state S of the cubical system there is a token τ which is effective for S with $\theta_\tau > 0$.

By applying Theorem 5.2, we can reformulate the result of the theorem in [18] as follows:

Theorem 8.1. *Let S_0 be a fixed state in $\mathcal{S}$. The function*

$$\nu(S) = \prod_{\tau \in \widehat{S} \setminus \widehat{S}_0} \frac{\theta_\tau}{\theta_{\tilde{\tau}}}$$

is an invariant measure satisfying the detailed balance equation

$$\nu(S)p(S,V) = \nu(V)p(V,S).$$

In particular, the chain $(\mathbf{S}_n)$ *is time reversible.*

We define

$$m(S) = \sum_{R\in\mathcal{S}} \prod_{\tau\in\widehat{R}\setminus\widehat{S}} \frac{\theta_\tau}{\theta_{\tilde{\tau}}},$$

assuming that $\prod_{\tau\in\varnothing}(\theta_\tau/\theta_{\tilde{\tau}}) = 1$. Clearly, $\sum_S \nu(S) = m(S_0)$. If $m(S_0) < \infty$, then we can normalize measure ν and obtain a stationary distribution

$$\pi(S) = \frac{1}{m(S_0)}\nu(S)$$

satisfying the detailed balance equation. By this equation and Theorem 5.2, we have

$$\pi(R) = \pi(S) \prod_{\tau\in\widehat{R}\setminus\widehat{S}} \frac{\theta_\tau}{\theta_{\tilde{\tau}}}.$$

Since π is a distribution, we obtain

$$1 = \sum_{R\in\mathcal{S}} \pi(R) = \pi(S)m(S),$$

so $m(S) < \infty$ and $\pi(S) = \frac{1}{m(S)}$ for all $S \in \mathcal{S}$.

The following theorem summarizes these properties of the chain $(\mathbf{S}_n)$.

Theorem 8.2. *The chain* $(\mathbf{S}_n)$ *is irreducible and aperiodic. If the series*

$$m(S) = \sum_{R\in\mathcal{S}} \prod_{\tau\in\widehat{R}\setminus\widehat{S}} \frac{\theta_\tau}{\theta_{\tilde{\tau}}}$$

converges for some state S, it converges for all states. In this case the chain $(\mathbf{S}_n)$ *is ergodic with a stationary distribution given by*

$$\pi(S) = \frac{1}{m(S)} = \left[\sum_{R\in\mathcal{S}} \prod_{\tau\in\widehat{R}\setminus\widehat{S}} \frac{\theta_\tau}{\theta_{\tilde{\tau}}}\right]^{-1} \qquad (8.1)$$

The distribution π satisfies the detailed balance equation

$$\pi(S)p(S,V) = \pi(V)p(V,S).$$

In particular, the chain $(\mathbf{S}_n)$ *is time reversible and $m(S)$ is the mean recurrence time for the state S.*

Note that the condition $m(S) < \infty$ is also necessary for the existence of a stationary distribution π.

If $\mathcal{S}$ is a finite set, then $m(S)$ is a finite function. It is easy to show that in this case (8.1) can be written in the form

$$\pi(S) = \frac{\prod_{\tau\in\widehat{S}} \theta_\tau}{\sum_{R\in\mathcal{S}} \prod_{\tau\in\widehat{R}} \theta_\tau}$$

(cf. [5]).

9 Conclusion

We have investigated algebraic and stochastic properties of cubical systems and shown that main results of media theory hold for cubical systems.

The structural properties of message and state contents (Theorem 5.2), together with the representation theorem (Theorem 6.1), reveal the binary nature of states in both media and cubical systems theories, which is also demonstrated by the 'cubical' structure of the corresponding graphs (Theorem 6.2). This characterization of states is crucial for the stochastic token theory (Theorem 8.2). Because any subgraph of a cube is a disjoint union of connected cubical graphs, it is appropriate to say that cubical systems represent the most general case of token systems enjoying the binary structure of their states.

Our treatment of cubical systems as token systems rather than transition systems is motivated by examples in Section 7 and connections with media theory. On the other hand, general methods of "concurrency" theory [19], and especially "geometric" models for concurrency [17, 9] could bring new elements to cubical token systems theory. In particular, a topological cubical complex can be associated with a cubical system in a natural way. Such complexes were used in the treatment of weak order families as media in [11, 12].

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