Axioms of Paraconsistency and Paraconsistent Consequence Relation

Mihir K. Chakraborty
Department of Pure Mathematics
University of Calcutta, India
mihirc99@vsnl.com

Soma Dutta
Department of Pure Mathematics
University of Calcutta, India
somadutta77@yahoo.co.in

Abstract

In this paper axioms of parainconsistency as well as those of paraconsistent consequence have been introduced and their equivalence established.

Keywords: Paraconsistent logic, logical consequence, explosiveness

1 Introduction

Paraconsistent logic systems have a long history. For an overview one is referred to [1]. This paper investigates into the axioms that are to be satisfied for a set of formulae to be parainconsistent. In the context of classical logic it is known what axioms a consistent set fulfils [3]. It is also known what axioms are satisfied by the consequence relation of classical logic [3]. The equivalence between the two sets of axioms is also established. So it seems natural to investigate into the axioms to be satisfied by the consequence relation of paraconsistent logic systems. Among the wide variety of such logics one common agreement is to do away with explosiveness, that is, the feature that if a set of well formed formulae entails a wff and its negation too, the set does necessarily entail all wffs. This is a property of classical logic. It is also natural to examine the interconnections between the axioms of paraconsistency and paraconsistent consequence. To put the task in more concrete terms, the main issue is to frame the two sets of axioms in such a way that they are equivalent as it is in the case of classical logic. In this paper two pairs of such sets of axioms are presented.

We have not delved into the philosophical significance of these axioms. This paper is basically mathematical in nature. However, some very interesting insights are thrown into the notions that are the fundamental ingredients of paraconsistent logic systems.

Let us first look into the two sets of axioms in the classical case.

<u>Classical consequence axioms</u>:

As presented in [3], classical notion of consequence is a function C: P(F) □ P(F) , where F is the set of wffs, and P(F) , the power set of F, assigning each set of formulae X to its consequence set C(X) satisfying,

C1. $X \subseteq C(X)$

C2. If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

C3. $C(C(X)) = C(X)$

C4. $C(X) = \bigcup_{Y \subseteq X,\, Y \text{ is finite}} C(Y)$

C5. $\alpha \supset \beta \in C(X)$ iff $\beta \in C(X \cup \{\alpha\})$

C6. $C(\{\alpha, \sim\alpha\}) = F$

C7. $C(\{\alpha\}) \cap C(\{\sim\alpha\}) = C(\Phi)$

<u>Classical consistency axioms</u>:

CONS is a unary relation over P(F) satisfying,

Cons1. If $X \subseteq Y$ then $Y \in$ CONS implies $X \in$ CONS

Cons2. If $X \notin$ CONS then for some finite subset Y of X, $Y \notin$ CONS

Cons3. If $X \in$ CONS then either $X \cup \{\alpha\} \in$ CONS or $X \cup \{\sim\alpha\} \in$ CONS

Cons4. $\{\alpha, \sim\alpha\} \notin$ CONS

Cons5. $X \cup \{\alpha, \sim\beta\} \in$ CONS iff $X \cup \{\sim(\alpha \supset \beta)\} \in$ CONS

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The two sets of axioms are equivalent via the following conversion theorems.

Theorems for conversion:

(i) Let C be a classical consequence operator, postulated by the axioms C1- C7. Let CONS be defined by X $\in$ CONS iff for any α, $\{\alpha, \sim\alpha\}$ is not included in C(X). Then CONS satisfies all the classical consistency axioms.

(ii) Let CONS be a unary relation satisfying classical consistency axioms, Cons1– Cons5. Let C be defined by, $\alpha \in C(X)$ iff $X \cup \{\sim\alpha\} \notin$ CONS. Then C satisfies all classical consequence axioms.

It is also established that if we start from C, obtain the CONS by (i) and then from CONS obtain C' by (ii) then $C = C'$. Similar is the case if instead of starting with C one starts with CONS.

We shall now observe that similar equivalence may be established by taking smaller subsets of the two groups of axioms. Particularly, the axioms dropped are related with compactness (C4 and Cons2) and implication (C5 and Cons5). In fact, we have wanted to arrive at the bare minimum. It should also be noted that we have taken inconsistency axioms now instead of consistency axioms. In the context of the present paper this change is made for mere convenience. But in our other works (as yet unpublished) switching over to inconsistency from consistency turns out to be essential.

Consequence axioms without implication and compactness

Let C: P(F) □ P(F) be a mapping, where F is the set of wffs, satisfying,

(C1′) $X \subseteq C(X)$

(C2′) If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

(C3′) $C(C(X)) = C(X)$

(C4′) $C(X\cup\{\alpha\}) \cap C(X\cup\{\sim\alpha\}) = C(X)$

(C5′) $C(\{\alpha, \sim\alpha\}) = F$

Inconsistency axioms without implication and compactness

Let INCONS be a unary relation over P(F) satisfying

Incons1. If $X \subseteq Y$ then $X \in$ INCONS implies $Y \in$ INCONS

Incons2. If $X \cup\{\sim\alpha\} \in$ INCONS, for each α in Y and $X \cup Y \in$ INCONS then $X \in$ INCONS

Incons3. $\{\alpha, \sim\alpha\} \in$ INCONS

It can be verified that these two modified sets of axioms also obey the above mentioned theorems for conversion.

2 An approach for axiomatization of a notion of non-explosive consequence:

2.1 It has been mentioned that one of the most basic features of the classical notion of consequence is its explosiveness i.e. a contradictory premise (a formula and its negation) yields every formula. In this regard paraconsistent logic is non-standard. A logic is paraconsistent if its notion of consequence is non-explosive i.e. it is not that for any α, $C(\{\alpha,\sim\alpha\})=F$ or in other words, there is some α, such that $C(\{\alpha,\sim\alpha\})\neq F$. From the definitions, mentioned above, it can be noticed that explosiveness condition is a formula-independent behaviour and in this sense, explosiveness condition of a consequence is global in character. Whereas non-explosiveness is a local property i.e. formula-dependent property of a consequence. For some α, $C(\{\alpha,\sim\alpha\})\neq F$ that means, there may exist some β, for which $C(\{\beta,\sim\beta\})=F$. Hence the behaviour of a non-explosive consequence depends on the formula, under consideration.

However, common sense would like to label both $\{\alpha, \sim\alpha\}$ and $\{\beta, \sim \beta\}$ as inconsistent sets. Usual practice in logic is to define its notion of inconsistency in terms of its consequence. Viz,

a set X is inconsistent if for some α, $\{\alpha, \sim \alpha\} \subseteq C(X)$ .... (1)

Note that in standard logics definition of inconsistency is equivalent to assuming $C(X) = F$. So it follows in classical case that if $\{\alpha, \sim \alpha\} \subseteq C(X)$ for some α, $\{\alpha, \sim \alpha\} \subseteq C(X)$ for all α.

Now, in the context of non-explosiveness let us consider two sets X and Y such that $\{\alpha, \sim \alpha\} \subseteq C(X)$ and $\{\beta, \sim\beta\} \subseteq C(Y)$, where $C(\{\alpha, \sim\alpha\}) \neq F$ and $C(\{\beta, \sim \beta\}) = F$

By monotonicity and idempotence, for any γ, $\{\gamma, \sim \gamma\} \subseteq C(Y)$ but in case of X, there may exist some δ, such that $\{\delta, \sim \delta\}$ is not included in $C(X)$

Thus if (1) is taken as the definition criterion for inconsistency then both X and Y are inconsistent but clearly nature of inconsistency differs.

It seems that for the set X, α determines the inconsistency of X and δ does not. This analysis again pushes us towards a notion, relativised with respect to formula.

In this regard our proposal for an adequate definition of inconsistency corresponding to a non-explosive notion of consequence , rather parainconsistency (a notion which is distinct from but similar to the notion of inconsistency) is ,

for a set X of formulae and a formula α, (X, α) is parainconsistent (i.e. X is inconsistent with respect to α) if $\{\alpha, \sim \alpha\} \subseteq C(X)$.

2.2 In 2.1 we discussed why at all we need to think parainconsistency as a relativised notion. Let us now think first what would be the plausible demands to a notion of parainconsistency and then find the corresponding notion of consequence.

<u>ParaINCONS_1 axioms</u>:

Let ParaINCONS_1 be a binary relation between P(F) and F, postulated by,

(PI_1)1. If $\alpha \in X$ then

$(X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$

(PI_1)2.If $X \subseteq Y$ then $(X, \alpha) \in \text{ParaINCONS}_1$ implies $(Y, \alpha) \in \text{ParaINCONS}_1$

(PI_1)3.If for all $\alpha \in Y$,

$(X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ then

$(X \cup Y, \beta) \in \text{ParaINCONS}_1$

implies $(X, \beta) \in \text{ParaINCONS}_1$

(PI_1)4. For some α, there is some β such that

$(\{\alpha, \sim\alpha\}, \beta) \notin \text{ParaINCONS}_1$

(PI_1)5. $(X \cup \{\alpha\}, \beta) \in \text{ParaINCONS}_1$ and

$(X \cup \{\sim\alpha\}, \beta) \in \text{ParaINCONS}_1$ imply

$(X, \beta) \in \text{ParaINCONS}_1$

(PI_1)6. $(X, \alpha) \in \text{ParaINCONS}_1$ iff

$(X, \sim\alpha) \in \text{ParaINCONS}_1$

(PI_1)7. $(X \cup \{\alpha\}, \beta) \in \text{ParaINCONS}_1$

iff $(X \cup \{\sim(\sim\alpha)\}, \beta) \in \text{ParaINCONS}_1$

As the name 'parainconsistency' suggests itself as a notion distinct from but similar to the notion of inconsistency, one can easily notice that the naturals demands to an inconsistent set are being relativised with respect to formula to get hold of the notion of parainconsistency. From classical standpoint, one can easily ascertain $X \cup \{\sim\alpha\}$as an inconsistent set if it is already known that α is a member of X. But, in this non-standard scenario, one can only ascertain the inconsistency of $X \cup \{\sim\alpha\}$with respect to α, when α is a member of X. Following the above line of thought, other postulates, imposed on ParaINCONS_1 can be justified easily. But incorporation of last two axioms needs to be especially justified.

In characterizing standard notion of inconsistency negation plays an important role but that turns out to be redundant on account of explosiveness. That is, whenever a set yields a formula and its negation, it yields every formula and conversely. But in determining parainconsistency, negation plays an essential role. So, in proposing ParaINCONS_1 , a formula-dependent notion, relation between the assertions 'X is inconsistent with respect to α' and 'X is inconsistent with respect to ~α' need to be specified. On the other hand (PI_1)7 asserts that inconsistency of $X \cup \{\alpha\}$ with respect to β implies inconsistency of $X \cup \{\sim \sim\alpha\}$ with respect to β and conversely. It is clear that (PI_1)7 does not follow from $(PI_1)6$, although this seems to be a natural expectation to the notion of parainconsistency. For the technical reason of proving the conversion theorems, given below, $(PI_1)7$ is explicitly required along with $(PI_1)6$.

As the notion of inconsistency and consequence are interwoven in classical logic, a similar connection is sought for this non-standard scenario too and this search gives rise to a notion of consequence C: P(F) □ P(F) , satisfying,

$(PC_1)1$. $X \subseteq C(X)$

$(PC_1)2$. If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

$(PC_1)3$. $C(C(X)) = C(X)$

$(PC_1)4$. For some α, $C(\{\alpha, \sim\alpha\}) \neq F$

$(PC_1)5$. $C(X \cup \{\alpha\}) \cap C(X \cup \{\sim\alpha\}) = C(X)$

$(PC_1)6$. $\alpha \in C(X)$ iff $\sim(\sim\alpha) \in C(X)$

$(PC_1)7$. $C(X \cup \{\alpha\}) = C(X \cup \{\sim(\sim\alpha)\})$

2.3 Theorems of conversion between non-explosive consequence and $ParaINCONS_1$

(i) Let C be an operator on P(F), satisfying $(PC_1)1$ to $(PC_1)7$. Define $ParaINCONS_1$ by, $(X, \alpha) \in ParaINCONS_1$ iff $\{\alpha, \sim\alpha\} \subseteq C(X)$. Then $ParaINCONS_1$ will satisfy all $(PI_1)1$ – $(PI_1)7$ axioms

(ii) Let $ParaINCONS_1$ be a binary relation between P(F) and F, satisfying all $(PI_1)1$-$(PI_1)7$ axioms. Define C by, $\alpha \in C(X)$ iff $(X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$ Then C turns out to be a consequence operator relative to $ParaINCONS_1$ i.e. C satisfies $(PC_1)1$-$(PC_1)7$.

Proof: (i)

$(PI_1)1$. Let $\alpha \in X$ then $\alpha \in X \cup \{\sim\alpha\}$
$\therefore\ \alpha \in C(X \cup \{\sim\alpha\})$ (by $(PC_1)1$).

Also $\sim\alpha \in X \cup \{\sim\alpha\}$ implies $\sim\alpha \in C(X \cup \{\sim\alpha\})$ (by $(PC_1)1$). $\therefore \{\alpha, \sim\alpha\} \subseteq C(X \cup \{\sim\alpha\})$

$\therefore (X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$

$(PI_1)2$. Let $X \subseteq Y$ and $(X, \alpha) \in ParaINCONS_1$

$\therefore \{\alpha, \sim\alpha\} \subseteq C(X) \subseteq C(Y)$.

$\therefore (Y, \alpha) \in ParaINCONS_1$

$(PI_1)3$. Let for all $\alpha \in Y$
$(X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$

$\therefore \{\alpha, \sim\alpha\} \subseteq C(X \cup \{\sim\alpha\})$.

Also $\alpha \in C(X \cup \{\alpha\})$, (by $(PC_1)1$)

As α belongs to both $C(X \cup \{\alpha\})$ and $C(X \cup \{\sim\alpha\})$, by (($PC_1)5$ $\alpha \in C(X)$, i.e

for all $\alpha \in Y$, $\alpha \in C(X)$ in other words, $Y \subseteq C(X)$

Also by $(PC_1)1$, $X \subseteq C(X)$

$\therefore\ X \cup Y \subseteq C(X)$

$\therefore\ C(X \cup Y) \subseteq C(C(X)) = C(X)$ by $(PC_1)2$ and $(PC_1)3$.

Now let $(X \cup Y, \beta) \in ParaINCONS_1$
$\therefore \{\beta, \sim\beta\} \subseteq C(X \cup Y) \subseteq C(X)$

$\therefore (X, \beta) \in ParaINCONS_1$

$(PI_1)4$. By $(PC_1)4$. we have, for some α, $C(\{\alpha, \sim\alpha\}) \neq F$ i.e. there is some β such that

$\beta \notin C(\{\alpha, \sim\alpha\})$ and hence $\{\beta, \sim\beta\} \not\subseteq C(\{\alpha, \sim\alpha\})$

$\therefore (\{\alpha, \sim\alpha\}, \beta) \notin ParaINCONS_1$

$(PI_1)5$. Let $(X \cup \{\alpha\}, \beta) \in ParaINCONS_1$ and $(X \cup \{\sim\alpha\}, \beta) \in ParaINCONS_1$
$\therefore \{\beta, \sim\beta\} \subseteq C(X \cup \{\alpha\})$ and
$\{\beta, \sim\beta\} \subseteq C(X \cup \{\sim\alpha\})$. Then by (($PC_1)5$,
$\{\beta, \sim\beta\} \subseteq C(X)$ i.e. $(X, \beta) \in ParaINCONS_1$

$(PI_1)6$. $(X, \alpha) \in ParaINCONS_1$ iff
$\{\alpha, \sim\alpha\} \subseteq C(X)$ iff $\{\sim\sim\alpha, \sim\alpha\} \subseteq C(X)$,
by $(PC_1)6$

$\therefore (X, \sim\alpha) \in ParaINCONS_1$

$(PI_1)7$. $(X \cup \{\beta\}, \alpha) \in ParaINCONS_1$ iff
$\{\alpha, \sim\alpha\} \subseteq C(X \cup \{\beta\}) = C(X \cup \{\sim\sim\beta\})$ iff
$(X \cup \{\sim\sim\beta\}, \alpha) \in ParaINCONS_1$ (by $(PC_1)7$)

(ii)

(PC$_1$)1. Let $\alpha \in X$
$\therefore (X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ (by (PI$_1$)1)

$\therefore\ \alpha \in C(X)$

(PC$_1$)2. Let $X \subseteq Y$

$\therefore\ X \cup \{\sim\alpha\} \subseteq Y \cup \{\sim\alpha\}$, for any α,
$\therefore (X \cup \{\sim\alpha\}, \beta) \in \text{ParaINCONS}_1$ implies
$(Y \cup \{\sim\alpha\}, \beta) \in \text{ParaINCONS}_1$ for any β,
$\therefore$ In particular for $\beta = \alpha$,
$(X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ implies
$(Y \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$
$\therefore\ \alpha \in C(X)$ implies $\alpha \in C(Y)$ i.e. $C(X) \subseteq C(Y)$

(PC$_1$)3. Let $Y \subseteq C(X)$ i.e. $\alpha \in Y$ implies $\alpha \in C(X)$
$\therefore (X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ (by definition)
$\therefore (X \cup \{\sim\beta\} \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$
(by (PI$_1$)2) i.e. for all $\alpha \in Y$
$(X \cup \{\sim\beta\} \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$
Then by (PI$_1$)3,
$(X \cup \{\sim\beta\} \cup Y, \gamma) \in \text{ParaINCONS}_1$ implies
$(X \cup \{\sim\beta\}, \gamma) \in \text{ParaINCONS}_1$
$\therefore$ In particular for $\gamma = \beta$,
$(X \cup \{\sim\beta\} \cup Y, \beta) \in \text{ParaINCONS}_1$ implies
$(X \cup \{\sim\beta\}, \beta) \in \text{ParaINCONS}_1$
i.e. $\beta \in C(X \cup Y)$ implies $\beta \in C(X)$
$\therefore C(X \cup Y) \subseteq C(X)$

[As $Y \subseteq C(X)$ implies $C(X \cup Y) \subseteq C(X)$ iff $C(C(X)) = C(X)$, instead of $C(C(X)) = C(X)$ we have proved $Y \subseteq C(X)$ implies $C(X \cup Y) \subseteq C(X)$]

(PC$_1$)4. By (PI$_1$)4. for some α, there is some β such that $(\{\alpha, \sim\alpha\}, \beta) \notin \text{ParaINCONS}_1$
By (PI$_1$)5. we have
$(\{\alpha, \sim\alpha\} \cup \{\gamma\}, \beta) \in \text{ParaINCONS}_1$ and
$(\{\alpha, \sim\alpha\} \cup \{\sim\gamma\}, \beta) \in \text{ParaINCONS}_1$ imply
$(\{\alpha, \sim\alpha\}, \beta) \in \text{ParaINCONS}_1$ for any γ.

In particular for $\gamma = \beta$,
$(\{\alpha, \sim\alpha\} \cup \{\beta\}, \beta) \in \text{ParaINCONS}_1$ and
$(\{\alpha, \sim\alpha\} \cup \{\sim\beta\}, \beta) \in \text{ParaINCONS}_1$ imply
$(\{\alpha, \sim\alpha\}, \beta) \in \text{ParaINCONS}_1$

But as $(\{\alpha, \sim\alpha\}, \beta) \notin \text{ParaINCONS}_1$ either
$(\{\alpha, \sim\alpha\} \cup \{\beta\}, \beta) \notin \text{ParaINCONS}_1$ or
$(\{\alpha, \sim\alpha\} \cup \{\sim\beta\}, \beta) \notin \text{ParaINCONS}_1$

Case-I $(\{\alpha, \sim\alpha\} \cup \{\sim\beta\}, \beta) \notin \text{ParaINCONS}_1$
$\therefore \beta \notin C(\{\alpha, \sim\alpha\})$

Case-II $(\{\alpha, \sim\alpha\} \cup \{\beta\}, \beta) \notin \text{ParaINCONS}_1$
implies $(\{\alpha, \sim\alpha\} \cup \{\sim\sim\beta\}, \beta) \notin \text{ParaINCONS}_1$
(by (PI$_1$)7)
implies $(\{\alpha, \sim\alpha\} \cup \{\sim\sim\beta\}, \sim\beta) \notin \text{ParaINCONS}_1$
(by (PI$_1$)6)
$\therefore \sim\beta \notin C(\{\alpha, \sim\alpha\})$
$\therefore$ From I and II we have $C(\{\alpha, \sim\alpha\}) \neq F$

(PC$_1$)5. Let $\beta \in C(X \cup \{\alpha\})$ and $\beta \in C(X \cup \{\sim\alpha\})$
$\therefore (X \cup \{\alpha\} \cup \{\sim\beta\}, \beta)) \in \text{ParaINCONS}_1$ and
$(X \cup \{\sim\alpha\} \cup \{\sim\beta\}, \beta)) \in \text{ParaINCONS}_1$

i.e. $(X \cup \{\sim\beta\} \cup \{\alpha\}, \beta) \in \text{ParaINCONS}_1$ and
$(X \cup \{\sim\beta\} \cup \{\sim\alpha\}, \beta) \in \text{ParaINCONS}_1$
$\therefore\ (X \cup \{\sim\beta\}, \beta) \in \text{ParaINCONS}_1$ (by (PI$_1$)5)

$\therefore \beta \in C(X)$

(PC$_1$)6. Let $\alpha \in C(X)$ iff
$(X \cup \{\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ iff
$(X \cup \{\sim\sim\sim\alpha\}, \alpha) \in \text{ParaINCONS}_1$ (by (PI$_1$)7)
iff $(X \cup \{\sim\sim\sim\alpha\}, \sim\alpha) \in \text{ParaINCONS}_1$
(by (PI$_1$)6) iff
$(X \cup \{\sim\sim\sim\alpha\}, \sim\sim\alpha) \in \text{ParaINCONS}_1$
(by (PI$_1$)6) iff $\sim\sim\alpha \in C(X)$

(PC$_1$)7. $\beta \in C(X \cup \{\alpha\})$ iff
$(X \cup \{\alpha\} \cup \{\sim\beta\}, \beta)) \in \text{ParaINCONS}_1$ iff
$(X \cup \{\sim\beta\} \cup \{\sim\sim\alpha\}, \beta)) \in \text{ParaINCONS}_1$
(by (PI$_1$)7) iff $\beta \in C(X \cup \{\sim\sim\alpha\})$

These two notions perfectly match with each other.

2.4 Theorem: (i) Let C_1 be given. Define ParaINCONS_1 in terms of C_1. Then the operator C_2, generated from ParaINCONS_1 coincides with C_1

(ii) Let $ParaINCONS_1$ be given. Define C in terms of $ParaINCONS_1$. This C again generates the same $ParaINCONS_1$.

Proof: (i) Let C_1 be given. Define $ParaINCONS_1$ by $(X, \alpha) \in ParaINCONS_1$ iff $\{\alpha, \sim\alpha\} \subseteq C_1(X)$ and then in terms of $ParaINCONS_1$ define C_2 by $\beta \in C_2(X)$ iff $(X \cup \{\sim\beta\}, \beta) \in ParaINCONS_1$

Now let $\beta \in C_1(X)$ $\therefore \beta \in C_1(X \cup \{\sim\beta\})$ by $(PC_1)2$
Again $\sim\beta \in C_1(X \cup \{\sim\beta\})$ by $(PC_1)1$
$\therefore \{\beta, \sim\beta\} \subseteq C_1(X \cup \{\sim\beta\})$
$\therefore (X \cup \{\sim\beta\}, \beta) \in ParaINCONS_1$
$\therefore \beta \in C_2(X)$ $\therefore C_1(X) \subseteq C_2(X)$

Conversely, let $\beta \in C_2(X)$
$\therefore (X \cup \{\sim\beta\}, \beta) \in ParaINCONS_1$
$\therefore \{\beta, \sim\beta\} \subseteq C_1(X \cup \{\sim\beta\})$
Also $\therefore \beta \in C_1(X \cup \{\beta\})$ by $(PC_1)1$
$\therefore \beta \in C_1(X \cup \{\beta\})$ and $\beta \in C_1(X \cup \{\sim\beta\})$ imply $\beta \in C_1(X)$ by $(PC_1)5$
$\therefore C_2(X) \subseteq C_1(X)$ i.e. $C_1(X) = C_2(X)$

(ii) Let $ParaINCONS_1$ be given. Define C by $\alpha \in C(X)$ iff $(X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$
In terms of C let us define $ParaINCONS_1'$ by $(X, \beta) \in ParaINCONS_1'$ iff $\{\beta, \sim\beta\} \subseteq C(X)$

Let $(X, \alpha) \in ParaINCONS_1$
$\therefore (X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$ by $(PI_1)2$
$\therefore \alpha \in C(X)$

Again $(X, \alpha) \in ParaINCONS_1$ implies $(X, \sim\alpha) \in ParaINCONS_1$ by $(PI_1)6$
$\therefore (X \cup \{\sim\sim\alpha\}, \sim\alpha) \in ParaINCONS_1$ by $(PI_1)2$
$\therefore \sim\alpha \in C(X)$ $\therefore \{\alpha, \sim\alpha\} \subseteq C(X)$
$\therefore (X, \alpha) \in ParaINCONS_1'$

$\therefore ParaINCONS_1 \subseteq ParaINCONS_1'$

Conversely let, $(X, \alpha) \in ParaINCONS_1'$
$\therefore \{\alpha, \sim\alpha\} \subseteq C(X)$
Now $\alpha \in C(X)$ implies $(X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$ and $\therefore \sim\alpha \in C(X)$ implies $(X \cup \{\sim\sim\alpha\}, \sim\alpha) \in ParaINCONS_1$ $\therefore (X \cup \{\sim\sim\alpha\}, \alpha) \in ParaINCONS_1$ (by $(PI_1)6$) $\therefore (X \cup \{\sim\alpha\}, \alpha) \in ParaINCONS_1$ and $(X \cup \{\sim\sim\alpha\}, \alpha) \in ParaINCONS_1$ imply $(X, \alpha) \in ParaINCONS_1$ $\therefore ParaINCONS_1' \subseteq ParaINCONS_1$

i.e. $ParaINCONS_1$ and $ParaINCONS_1'$ coincide with each other.

3 Another approach for axiomatization of a non-explosive notion of consequence with an operator ′:

3.1 As is mentioned earlier, to axiomatize $ParaINCONS_1$ we need to specify $(X, \alpha) \in ParaINCONS_1$ iff $(X, \sim\alpha) \in ParaINCONS_1$ which immediately implies $(X, \alpha) \in ParaINCONS_1$ iff $(X, \sim\sim\alpha) \in ParaINCONS_1$. Thus $(PI_1)6$ together with $(PI_1)7$ demand a sort of similar treatment to a formula and its double negation. In this new approach we present an alternative axiomatization so that there may be a case where X is inconsistent with respect to α but not with respect to $\sim\sim\alpha$.

3.2 Definition: Let ′ be a function from F to F, defined by,

$\alpha' = \beta$, if $\alpha = \sim\beta$, for some β

$= \sim\alpha$, otherwise

We now define a different notion of Parainconsistency.

3.3 Axioms of $ParaINCONS_2$ and the relative notion of consequence with respect to the operator ′:

$ParaINCONS_2$ axioms:

Let $ParaINCONS_2$ be a binary relation between P(F) and F, postulated by,

$(PI_2)1$. If $\alpha \in X$ then

$(X \cup \{\alpha'\}, \alpha) \in ParaINCONS_2$

$(PI_2)2$. If $X \subseteq Y$ then $(X, \alpha) \in ParaINCONS_2$ implies $(Y, \alpha) \in ParaINCONS_2$

$(PI_2)3$. If for all $\alpha \in Y$,

$(X \cup \{\alpha'\}, \alpha) \in ParaINCONS_2$ then $(X \cup Y, \beta) \in ParaINCONS_2$ implies $(X, \beta) \in ParaINCONS_2$

$(PI_2)4$. For some α, there is some β such that

$(\{\alpha, \alpha'\}, \beta) \notin ParaINCONS_2$

$(PI_2)5.\ (X\cup\{\alpha\}, \beta) \in ParaINCONS_2$ and

$(X\cup\{\alpha'\}, \beta) \in ParaINCONS_2$ imply

$(X, \beta) \in ParaINCONS_2$

$(PI_2)6.\ (X, \alpha) \in ParaINCONS_2$ iff

$(X, \alpha') \in ParaINCONS_2$

$(PI_2)7.\ (X\cup\{\alpha\}, \beta) \in ParaINCONS_2$ iff

$(X\cup\{(\alpha')'\}, \beta) \in ParaINCONS_2$

Axioms for the notion of consequence correspondence to $ParaINCONS_2$:

Let C: P(F) □ P(F) be a mapping, where F is the set of wffs, satisfying,

$(PC_2)1.\ X \subseteq C(X)$

$(PC_2)2.$ If $X \subseteq Y$ then $C(X) \subseteq C(Y)$

$(PC_2)3.\ C(C(X)) = C(X)$

$(PC_2)4.$ For some α, $C(\{\alpha, \alpha'\}) \neq F$

$(PC_2)5.\ C(X\cup\{\alpha\}) \cap C(X\cup\{\alpha'\}) = C(X)$

$(PC_2)6.\ \alpha \in X$ iff $(\alpha')' \in X$

$(PC_2)7.\ C(X\cup\{\alpha\}) = C(X\cup\{(\alpha')'\})$

3.4 Theorems of conversion

(i) Let C be an operator on P(F), satisfying $(PC_2)1$-$(PC_2)7$. Define $ParaINCONS_2$ by

$(X, \alpha) \in ParaINCONS_2$ iff $\{\alpha, \alpha'\} \subseteq C(X)$. Then

$ParaINCONS_2$ will satisfy all $(PI_2)1$ - $(PI_2)7$ axioms

(ii)Let $ParaINCONS_2$ be a binary relation between P(F) and F, satisfying all $(PI_2)1$-$(PI_2)7$ axioms. Define C by, $\alpha \in C(X)$ iff $(X\cup\{\alpha'\}, \alpha) \in ParaINCONS_2$. Then C turns out to be a consequence operator relative to $ParaINCONS_2$ i.e. C satisfies $(PC_2)1$-$(PC_2)7$ axioms.

Note: $ParaINCONS_2$ and the relative notion of consequence also match with each other in the sense, mentioned in section 4.

4 Inter-relation between the two notions of inconsistency, mentioned above:

Let us start with an operator C, called a consequence operator, assigning each set of formulae X to a set of formulae C(X), known as the consequence set of X. In terms of the operator C (without imposing any condition on C) let the following relations be defined.

1. A unary relation INCONS over P(F) is defined by, $X \in$ INCONS iff for some α,

$\{\alpha, \sim\alpha\} \subseteq C(X)$.

Note: It does not mean that automatically $\{\alpha, \sim\alpha\}$ is a subset of C(X) for every α. For this to hold further axioms on C are required.

2. A binary relation $ParaINCONS_1$ between P(F) and F is defined by, $(X, \alpha) \in ParaINCONS_1$

iff $\{\alpha, \sim\alpha\} \subseteq C(X)$.

3. A binary relation $ParaINCONS_2$ between P(F) and F is defined by, $(X, \alpha) \in ParaINCONS_2$ iff $\{\alpha, \alpha'\} \subseteq C(X)$, where $'$ is a function from F to F, defined by,

$\alpha' = \beta$, if $\alpha = \sim\beta$, for some β

$= \sim\alpha$, otherwise

Observations:

(1) For any α, $(X, \alpha) \in ParaINCONS_1$ implies $X \in$ INCONS
(2) If $X \in$ INCONS there is some α such that $(X, \alpha) \in ParaINCONS_1$
(3) For any α, $(X, \alpha) \in ParaINCONS_2$ implies $X \in$ INCONS
(4) If $X \in$ INCONS there is some α such that $(X, \alpha) \in ParaINCONS_2$
(5) $(X, \sim\beta) \in ParaINCONS_1$ implies $(X, \sim(\sim\beta)) \in ParaINCONS_2$
(6) $(X, \sim\beta) \in ParaINCONS_2$ implies $(X, \beta) \in ParaINCONS_1$
(7) Combining (5) and (6), $(X, \sim\beta) \in ParaINCONS_1$ iff $(X, \sim(\sim\beta)) \in ParaINCONS_2$
(8) $(X, \sim\beta) \in ParaINCONS_2$ iff $(X, \beta) \in ParaINCONS_1$, if β is of the form $\sim\gamma$, for some γ, (by (6) and (5))
(9) If α is not of the form $\sim\beta$, then $(X, \alpha) \in ParaINCONS_1$ iff $(X, \alpha) \in ParaINCONS_2$

5 Conclusion:
In this introductory endeavour, a first attempt is made to axiomatize the notions of inconsistency and consequence of paraconsistent logic systems. It is obvious that not all such systems possess these properties. We are thankful to Graham Priest for mentioning this point through a personal communication. We know that there are paraconsistent systems that are not monotonic or that do not satisfy the cut rule. The core property that has been touched upon in this work is non-explosiveness. Keeping this core intact what other categorizations are possible should be our future attempts.

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